Exact classical matching and certified pole reconstruction in a Gaussian extension of a two-field model

Josip Fellner
Independent Researcher
ORCID: 0009-0003-1705-3985
info@apeironfield.com

P01 Supplement S3
Publication edition 1.0-publication.1 — 25 September 2026

Abstract

A finite inverse bound for a truncated two-field dispersion relation requires a bound on the omitted dynamics. We construct that input for an explicitly declared classical extension: a massive Gaussian scalar coupled to an affine radial function times the phase kinetic invariant, with a specified local matching subtraction. The rolling background remains an exact solution. Eliminating the extra field gives a rank-one rational correction, whose full frequency dependence fixes an analytic domain and a geometric remainder without estimating unknown Wilson coefficients. The complete quadratic Hamiltonian provides a sufficient stability condition. Separate light and heavy pole bounds distinguish the inferred reference gap from the actual heavy frequencies of the three-field system. We combine the resulting model error with simultaneous measurement errors in a finite three-node reconstruction test. The construction is a conditional model extension, not a derived ultraviolet completion or a bound on omitted quantum contributions. Existing frozen-vacuum loop residues do not supply those missing physical inputs.

1 Question and status of the model

The reference problem reconstructs three spectral parameters from three positions of the lower pole of a specified classical two-field system. Its finite-error extension accepts a separately justified model discrepancy. The question here is whether an explicit action can supply that discrepancy rather than an arbitrary kernel envelope.

We use the type of linear derivative source found in the project's earlier conditional Gaussian-mediator ansatz. We add a local subtraction as an explicit matching choice. This new choice is essential: the original ansatz has a different action and a constant vacuum, whereas the reference inverse problem has a rolling background. No identification of their numerical parameters or physical observables is made. The original derivative determinant interaction remains an independent part of the reference action.

The forward unequal-speed two-field propagator is already known [1]. Bounded-error inverse estimation [2] and analytic eigenvalue localization [3] are also established methods. Our task is the explicit classical action-to-remainder calculation and its finite inverse application, not a new general spectral or estimation algorithm.

The result is parametric. The mediator mass and couplings are inputs, or must belong to an independently justified set. The analytic domain derived below is a domain of the exact quadratic calculation and of its derivative expansion. It is not, by itself, a quantum or nonlinear ultraviolet cutoff. The model includes a nonrenormalizable derivative source; integrating its Gaussian field does not make the complete interacting quantum theory Gaussian.

2 Reference action and inverse data

In signature $(+---)$ with a fixed external metric, let

$$Y_s = (\partial\sigma)^2, \qquad Y_t = (\partial\theta)^2, \qquad Z = \partial\sigma \cdot \partial\theta.$$

The reference Lagrangian is

$$\mathcal{L}_C = \frac{1}{2}Y_s + \frac{1}{2}(1+g\sigma)Y_t - \frac{1}{2}gv^2\sigma - \frac{1}{2}m^2\sigma^2 + C(Z^2 - Y_sY_t). \tag{1}$$

It admits $\sigma_0 = 0, \theta_0 = vt$. For $\sigma = s, \theta = vt + \pi$, define

$$D = gv, \quad M = m^2, \quad N = D^2, \quad B = M + N, \quad r = M/B, \quad a = 1 - 2Cv^2.$$

We require $a > 0$, $M > 0$ and $N > 0$. With $x = k^2$ and $y = \omega^2$,

$$P_0 = \begin{pmatrix} y - ax - M & -iD\omega \\ iD\omega & y - x \end{pmatrix}, \tag{2}$$

$$F_0 = \det P_0 = y^2 - [B + (a+1)x]y + ax^2 + Mx, \tag{3}$$

$$y_\pm = \frac{B + (a+1)x \pm G(x)}{2}, \qquad G(x) = \sqrt{B^2 + 2B(a+1-2r)x + (a-1)^2x^2}. \tag{4}$$

For $x > 0$, $0 < y_- < x < y_+$. The reference heavy squared gap is B. The calibrated metric and frequency normalization are fixed throughout.

For three distinct positive nodes, set $z_i = y_i/x_i$. The exact light data obey

$$x_i(1 - z_i)(a - z_i) = Bz_i - M.$$

Thus the rows $[x_i(1 - z_i), -z_i, 1]$ determine (a, B, M) linearly. The data matrix is invertible in the interior except at $a = r$. At that surface $y_- = rx$ and the light pole positions contain no information about B. This known reference-model exception is retained; the new action does not justify applying a three-parameter fit without a model-error bound.

3 A conditional classical Gaussian matching model

The following model is a new, explicitly selected classical extension of P01, motivated by the conditional Gaussian derivative-source mechanism discussed in P04 and its source audit. Those sources do not select a unique mediator for the P01 rolling background. No empirical coefficient, physical cutoff, full quantum matching result or ultraviolet completion is inferred. In particular, the added exchange does not derive or saturate P01's determinant-operator coefficient C.

3.1 Action, background and quadratic positivity

Set $\mathcal{L}_0 = \mathcal{L}_C$ in Eq. (1), with $M = m^2$, and define the enlarged action by

$$\boxed{\mathcal{L} = \mathcal{L}_0 + \tfrac{1}{2}(\partial S)^2 - \tfrac{1}{2}LS^2 + SJ - \frac{J^2}{2L}, \qquad J = \tfrac{1}{2}(h_0 + h_1\sigma)Y_t, \quad L = M_S^2 > 0.} \tag{5}$$

Thus L is a mass squared throughout. The dimensions are $[h_0] = -1$, $[h_1] = -2$, $[J] = 3$, $[v] = 2$ and $[L] = [M] = [B] = 2$. The scalar S is even under theta reflection. The action preserves theta shifts, theta reflection, Lorentz symmetry and spatial parity. Absence of S self-interactions and light-dependent S^2 terms is part of the selected Gaussian class. The finite subtraction $-J^2/(2L)$ is a model/matching choice, not a counterterm value proved by the original EFT.

The original rolling configuration remains a solution with

$$\sigma_0 = 0, \quad \theta_0 = vt, \quad J_0 = h_0 v^2/2, \quad S_0 = J_0/L. \tag{6}$$

Indeed the added non-kinetic term is $-L(S - J/L)^2/2$, whose value and field and metric first variations vanish there, while S_0 is constant. The original scalar equations and rolling stress are therefore unchanged. Equality of all homogeneous off-shell histories is not asserted.

For $\sigma = s$, $\theta = vt + \pi$, $S = S_0 + H$, put

$$A = h_1 v^2/2, \qquad b = h_0 v, \qquad J_1 = As + b\dot{\pi}, \qquad c = 1 - b^2/L, \quad d = D - Ab/L. \tag{7}$$

Here $[A] = 2$, $[b] = 1$. The quadratic source term $J_2 = h_0(\partial\pi)^2/2 + h_1 v s\dot{\pi}$ cancels from the added quadratic action through cancellation of the $J_0 J_2$ terms. The result is

$$\begin{aligned}\mathcal{L}_2 = {} & \tfrac{1}{2}\dot{s}^2 + \tfrac{1}{2}c\dot{\pi}^2 + \tfrac{1}{2}\dot{H}^2 - \tfrac{1}{2}[a(\nabla s)^2 + (\nabla\pi)^2 + (\nabla H)^2] \\ & - \tfrac{1}{2}(M + A^2/L)s^2 - \tfrac{1}{2}LH^2 + AHs + ds\dot{\pi} + bH\dot{\pi}.\end{aligned} \tag{8}$$

Positive-Hamiltonian proposition. For $a, M, L > 0$ and $b^2 < L$, with arbitrary real A, the quadratic Hamiltonian is positive at nonzero momentum:

$$\begin{aligned}\mathcal{H}_2 = {} & \tfrac{1}{2}p_s^2 + \frac{(p_\pi - ds - bH)^2}{2c} + \tfrac{1}{2}p_H^2 \\ & + \tfrac{1}{2}[a(\nabla s)^2 + (\nabla\pi)^2 + (\nabla H)^2] + \tfrac{1}{2}Ms^2 + \tfrac{1}{2}L(H - As/L)^2.\end{aligned} \tag{9}$$

This follows by the Legendre transform. At zero momentum there is the expected shift mode. The boundary $b^2 = L$ has a degenerate Legendre map and is excluded. The assertion concerns local classical quadratic stability, not global nonlinear, quantum or gravitational stability.

3.2 Exact elimination and a rank-one remainder

Let $x = k^2$, $y = \omega^2$. In the P01 Fourier convention,

$$P_3 = \begin{pmatrix} y - ax - M - A^2/L & -id\omega & A \\ id\omega & cy - x & ib\omega \\ A & -ib\omega & y - x - L \end{pmatrix}. \tag{10}$$

For $L + x - y \neq 0$, its Schur complement is exactly

$$P_{\text{eff}} = P_0 + fW, \qquad P_0 = \begin{pmatrix} y - ax - M & -iD\omega \\ iD\omega & y - x \end{pmatrix}, \tag{11}$$

$$f = \frac{y - x}{L(L + x - y)}, \qquad W = \begin{pmatrix} A^2 & -iAb\omega \\ iAb\omega & b^2 y \end{pmatrix}. \tag{12}$$

The light block before elimination is $P_0 - W/L$, and its mixing with H is $(A, ib\omega)^T$, proving the formula. The matrix W has rank at most one: $\det W = 0$ identically. In particular,

$$F_{\text{eff}} = F_0 + fQ,$$

$$F_0 = y^2 - [B + (a+1)x]y + ax^2 + Mx, \tag{13}$$

$$Q = A^2(y - x) + b^2 y(y - ax - M) - 2DAb\, y. \tag{14}$$

There is no quadratic-in-remainder contribution. The dimensions of f, W, Q are $-2, 4, 6$, so the kernel remainder and determinant remainder have dimensions two and four. The complete polynomial identity is

$$\det P_3 = (y - x - L)F_0 - \frac{y - x}{L}Q. \tag{15}$$

It remains finite at the eliminated-propagator pole, which is not generally a physical eigenpole. Exact elimination requires a specified Green function or boundary prescription and must not discard independently excited heavy homogeneous solutions; the full polynomial retains the coupled spectrum.

3.3 Which parameters are matched

The subtraction matches the zeroth derivative term of the mediator propagator. It does not automatically produce a fully canonically normalized low-frequency local reference. With $s_y = y-x$, $f = s_y/L^2+s_y^2/L^3+\cdots$. For $A \neq 0$, the ss remainder therefore contains the two-derivative term $A^2(y-x)/L^2$, because the background linear source includes the algebraic fluctuation As.

If one instead matches through two fluctuation derivatives and canonically normalizes the radial field, the reference becomes

$$Z_s = 1 + A^2/L^2, \qquad s_c = \sqrt{Z_s}\, s, \tag{16}$$

$$a_c = \frac{a + A^2/L^2}{Z_s}, \qquad M_c = \frac{M}{Z_s}, \qquad D_c = \frac{D}{\sqrt{Z_s}}, \tag{17}$$

$$B_c = B/Z_s, \qquad r_c = r. \tag{18}$$

Pi's intrinsic speed remains one at this derivative order; the off-diagonal and pi-pi remainders begin at three and four derivatives, respectively. Source normalizations must transform with the field. The exact remainder after this congruence and local subtraction is no longer the rank-one formula (12). The present bounds consistently use the zeroth-propagator-order convention of that formula. Neither convention changes the calibrated metric. Without the explicit subtraction, a pure b source adds a leading pi time kinetic term and changes its intrinsic speed squared to $1/(1 + b^2/L)$, outside the original fixed-speed three-parameter reference unless treated explicitly.

In the selected reference convention the full light pole starts as

$$y_{\text{light}} = rx + \left[\frac{(1-r)(a-r)}{B} + \frac{(1-r)(DA+Mb)^2}{B^2L^2}\right] x^2 + O(x^3). \tag{19}$$

Thus a P01 jet inversion without the model-discrepancy bound does not generally recover the declared reference parameters. The error order here is a consequence of this chosen source and matching prescription, not a universal EFT assumption.

3.4 A whole-disk finite pole certificate

Fix a positive reference pole $y_0 = y_-(x)$ or $y_+(x)$, with reference gap $g = |y_+ - y_-| > 0$. Choose a disk $|y - y_0| \leq \rho$ satisfying

$$0 < \rho < \min(y_0, g/2), \qquad \eta_H = |L + x - y_0| - \rho > 0. \tag{20}$$

Set

$$Y = y_0 + \rho, \qquad T = |y_0 - x| + \rho, \qquad U = |y_0 - ax - M| + \rho, \tag{21}$$

$$Q_{\max} = A^2T + b^2YU + 2|DAb|Y, \qquad \epsilon_F = \frac{TQ_{\max}}{L\eta_H}. \tag{22}$$

The exact rational remainder $R = fQ$ is analytic and Schwarz-real on the entire disk and obeys $|R| \leq \epsilon_F$. The estimate controls the full disk, not just the unperturbed mass shell, and uses the exact rank-one determinant structure.

Finite pole-tube theorem. If $\epsilon_F < \rho(g-\rho)$, there is exactly one positive real simple full-system pole in that disk, and

$$|y_{\text{full}} - y_0| \le \frac{2\epsilon_F}{g + \sqrt{g^2 - 4\epsilon_F}} < \rho. \tag{23}$$

On the boundary, $|F_0| \ge \rho(g-\rho) > |R|$, so Rouché's theorem gives one zero counted with multiplicity. Reality and conjugation symmetry make it real; $\rho < y_0$ makes it positive. The mediator block is invertible in the disk. Finally, for $t = |y_{\text{full}} - y_0| < \rho < g/2$, its equation gives $t(g-t) \le \epsilon_F$, proving (23).

For the lower reference pole, one may impose additionally $\rho < x - y_0$ to identify the certified zero directly as the physical light root. There is exactly one full-model root in $0 < y < x$: positivity of (9) gives positive real squared frequencies, continuous in x; near zero one root has slope $r < 1$ and two are gapped; and $\det P_3(x,x) = LD^2x > 0$ forbids crossing the line $y = x$ for $x > 0$. The zero at $x = 0$ is treated directly, not by a disk required to have $\rho < y_0 = 0$. The useful on-shell identity

$$Q(y_0, x) = -\frac{[A(x-y_0) + Db\, y_0]^2}{x - y_0} \le 0 \tag{24}$$

does not replace the entire-disk majorant.

3.5 Separate heavy-gap control

At $x = 0$, let

$$c = 1 - b^2/L > 0, \quad K = (A - Db)^2, \quad T_H = L + cB + K/L. \tag{25}$$

Besides the shift root, the full heavy squared gaps solve

$$\boxed{cy^2 - T_H y + LB = 0.} \tag{26}$$

If $L > B$, the two roots are distinct and positive:

$$Y_H = \frac{2LB}{T_H + \sqrt{T_H^2 - 4cLB}}, \qquad Y_S = \frac{T_H + \sqrt{T_H^2 - 4cLB}}{2c}, \tag{27}$$

with $0 < Y_H \le B < L \le L/c \le Y_S$. The discriminant is $(L - cB)^2 + 2K(L + cB)/L + K^2/L^2 > 0$. Equality $Y_H = B$ holds exactly for $K = 0$, when $Y_S = L/c$. Thus the reference-heavy continuation and the extra heavy mode must be distinguished; neither gap is generally $\sqrt{L}$.

Writing $\delta_H = B - Y_H$, the exact equation and a finite bound are

$$\delta_H(L - cY_H) = KY_H/L, \qquad 0 \le \delta_H \le \frac{KB}{L(L - cB)} = \overline{\delta}_H. \tag{28}$$

If $\overline{\delta}_H < B$, the frequency-gap error is at most

$$|\sqrt{Y_H} - \sqrt{B}| \le \frac{\overline{\delta}_H}{\sqrt{B} + \sqrt{B - \overline{\delta}_H}}. \tag{29}$$

For a prescribed squared-gap tolerance $0 < \tau < B$, the exact criterion is

$$\delta_H \le \tau \quad \Longleftrightarrow \quad K \le \frac{L\tau[L - c(B - \tau)]}{B - \tau}. \tag{30}$$

This follows by solving the first identity in (28) for K; the resulting function of δ_H is strictly increasing on $[0, B)$.

A separate complex heavy tube centered at B may use $0 < \rho_H < \min(B/2, L - B)$ and the sharper bound

$$\epsilon_H = \frac{(B + \rho_H)^2[K + b^2\rho_H]}{L(L - B - \rho_H)}. \tag{31}$$

Indeed $Q(y, 0) = y[K + b^2(y - B)]$. If $\epsilon_H < \rho_H(B - \rho_H)$, the pole-tube theorem applies with $g = B$. The exact formulas above need no such sufficient smallness test and may be sharper. A light-frequency domain does not supply the required heavy-frequency domain. When B itself is reconstructed, combine its certified error with a heavy-shift bound taken uniformly over the justified joint parameter set.

3.6 Uniform windows, finite truncation, and the missing inputs

A uniform use of the theorem requires an independently justified candidate set with positive kinetic margin $1 - b^2/L \geq c_{\min} > 0$, positive domain bounds and, for the ordered heavy result, a positive $L - B$ margin. The pole centers, gaps, whole disks and mediator denominator distances must all be controlled on that set. Suprema of (22) supply classical pole-error sets to compose with measurement errors in the previous finite reconstruction theorem. Candidate-dependent bounds may instead remain explicit in set membership. One fitted-point evaluation is not a uniform certificate. The common window must satisfy both momentum and frequency conditions, as well as the inverse target-accuracy conditions. Any further physical cutoff, omitted operators and loops require separate inputs.

For $s_y = y - x$ and integer $n \geq 0$, the exact geometric identity is

$$f = \sum_{j=1}^{n} \frac{s_y^j}{L^{j+1}} + \frac{s_y^{n+1}}{L^{n+1}(L - s_y)}. \tag{32}$$

On a whole domain $|s_y| \leq \chi L$, $0 \leq \chi < 1$,

$$|f - f_n| \leq \frac{\chi^{n+1}}{L(1 - \chi)}, \qquad \|E - E_n\| \leq |f - f_n|\,\|W\|, \qquad |F_{\text{eff}} - F_n| \leq |f - f_n|\,|Q|. \tag{33}$$

The determinant formula remains linear because $E_n = f_n W$ is rank one. Bounds on W, Q are indispensable. Roots of finite higher-derivative truncations outside the certified domain are not physical conclusions about the full model.

Why a coupling bound is necessary for uniform heavy closeness. Fix admissible P01 parameters, $L > B$ and $b^2 < L$, and let $|A| \to \infty$. The quadratic Hamiltonian remains positive and the same geometric-resolvent domain is available, but $K = (A - Db)^2 \to \infty$ and

$$Y_H \sim L^2 B/K \longrightarrow 0, \qquad Y_S \sim K/(cL) \longrightarrow \infty. \tag{34}$$

Thus the subtraction, positivity and resolvent domain alone do not uniformly keep the physical heavy gap near B. A controlled coupling set or source-sensitive information such as (30) is required. This is a scoped family within the selected model, not an all-method or all-completion impossibility claim. The comparison uses the chosen zeroth-propagator-order reference B; in the alternative convention (18), B_c itself tends to zero.

Some special limits sharpen the scope. $A = b = 0$ gives P01 plus a decoupled scalar. $A = Db$ protects the reference heavy gap at zero momentum without necessarily removing light corrections. If $a = r$ and $DA + Mb = 0$, the full light branch is exactly $y = rx$, even for nonzero extension couplings. This proves preserved linearity and permits a tuned family when those

couplings can vary accordingly; it does not prove a free B ambiguity when A, b are separately fixed and known, because the tuning relation may constrain B. The P01 reference matrix is still singular at $a = r$. Decoupling as $L \to \infty$ must specify how A, b scale. The boundary $b^2 = L$ and disks touching the Schur denominator remain outside their stated hypotheses.

Beyond P01, the missing inputs are the physically selected source and finite matching convention, L, A, b or a justified joint bounded set, consistent field/source and rolling-background normalization, and any further quantum or omitted-operator errors and empirical measurement set. P01 pole data determine D^2, not the relative orientation entering DAb; the invariant combinations $(A - Db)^2$ and $(DA + Mb)^2$ must be retained. With calibrated $v \neq 0$, $h_0 = b/v$ and $h_1 = 2A/v^2$. No globally minimal observable count for the enlarged inverse problem is claimed.

Integrating the Gaussian scalar at fixed light background and flat metric gives the exact tree functional $J[(L+\Box)^{-1} - 1/L]J/2$ and a light-independent pure-heavy determinant. It does not eliminate mixed heavy/light quantum contributions, light loops or finite renormalization data. The derivative source is itself nonrenormalizable by power counting. The conclusion is therefore an exact conditional classical model and its certified quadratic matching error, with quantum and empirical completion left explicit.

Verification. The standard-library code retained in the internal research archive verifies 22 symbolic identities with exact sparse multivariate polynomials and rational coefficients, including background cancellation, the full determinant, rank-one structure, heavy and infrared projections and finite geometric remainders. No old test suite, parameter scan or numerical physical example is used. The inequalities and domain statements are proved above; the code checks the algebra rather than replacing those proofs. No AP1 gate or uncomputed physical response is changed.

4 A finite action-to-inverse application

This section gives a nonempty mathematical design set for the selected classical quadratic model. The numbers are synthetic assumptions, not measurements, physically derived coupling bounds, or a physical cutoff. The same independently specified parameter set controls the action remainder, the light poles, and the inverse certificate. The inferred parameters use the zeroth-propagator-order reference P_0 defined above.

4.1 Finite inverse interface and proof

Fix a reference mass unit μ before fitting. In this section all quantities are numerical values in units of μ: in particular $x = k^2/\mu^2$, $y = \omega^2/\mu^2$, and B, M, L, A denote their mass-squared values divided by μ^2, while D, b are divided by μ. Take three exactly calibrated positive nodes x_i. Let y_i be their reference P01 lower roots, and suppose the observations obey $|\widehat{y}_i - y_i| \leq e_i$. Define

$$\begin{aligned} \widehat{z}_i &= \widehat{y}_i/x_i, & \widehat{\mathcal{A}}_{i\bullet} &= (x_i - \widehat{y}_i, -\widehat{y}_i/x_i, 1), & \widehat{f}_i &= \widehat{y}_i(1 - \widehat{y}_i/x_i), \\ \widehat{p} &= \widehat{\mathcal{A}}^{-1}\widehat{f}, & p &= (a, B, M)^T. \end{aligned} \tag{35}$$

The matrix must be invertible. With the induced infinity norm, put

$$\begin{aligned} E_{\mathcal{A}} &= \max_i e_i(1 + x_i^{-1}), & E_f &= \max_i \left(|1 - 2\widehat{z}_i|e_i + e_i^2/x_i\right), \\ \mathcal{N} &= \|\widehat{\mathcal{A}}^{-1}\|_\infty, & \alpha &= \mathcal{N}E_{\mathcal{A}} < 1, \\ E_p &= \frac{\mathcal{N}(E_f + E_{\mathcal{A}}\|\widehat{p}\|_\infty)}{1 - \alpha}. \end{aligned} \tag{36}$$

Then $\|p-\widehat{p}\|_\infty \le E_p$. Sufficient strict admissibility and true-relative accuracy conditions are

$$\widehat{a} > E_p, \qquad \widehat{M} > E_p, \qquad \widehat{B} - \widehat{M} > 2E_p, \tag{37}$$

$$\frac{|B-\widehat{B}|}{B} \le \frac{E_p}{\widehat{B}-E_p}, \qquad |r-\widehat{r}| \le R_r := \frac{E_p(1+|\widehat{r}|)}{\widehat{B}-E_p}, \quad \widehat{r} = \widehat{M}/\widehat{B}. \tag{38}$$

These statements are conditional on a common admissible model and its error set; three fitted observations alone do not establish model adequacy.

For completeness, dividing $F_0(y_i, x_i) = 0$ by x_i gives $\mathcal{A}p = f$ with the unperturbed versions of (35). The row perturbation is bounded by $e_i(1 + x_i^{-1})$. Expanding $f_i = y_i - y_i^2/x_i$ around $\widehat{y}_i$ gives precisely the bound E_f. The Neumann bound $\|\mathcal{A}^{-1}\|_\infty \le \mathcal{N}/(1-\alpha)$, applied to

$$\mathcal{A}(p-\widehat{p}) = (f-\widehat{f}) + (\widehat{\mathcal{A}}-\mathcal{A})\widehat{p},$$

proves (36). Positivity of the intervals proves (37); using $B \ge \widehat{B} - E_p > 0$ and expanding the ratio M/B proves (38).

4.2 An independent joint parameter set

Set $v = 1$ in the fixed units. Specify the set Θ before generating or fitting data:

$$\begin{aligned} &a \in [0.79, 0.81], \quad B \in [1.99, 2.01], \quad M \in [0.99, 1.01], \\ &A \in [0.001, 0.003], \quad b \in [0.0005, 0.0015], \quad L \in [99, 101], \\ &D^2 = B - M. \end{aligned} \tag{39}$$

Both signs of D are included. All intervals except the displayed relation are independent, and their endpoints are exact decimal rationals. Thus $0.9 < |D| < 1.1$, and usable conservative bounds are

$$r_- := \frac{0.99}{2.01} = \frac{33}{67} \le r \le r_+ := \frac{1.01}{1.99} = \frac{101}{199} < a_- = 0.79.$$

Write $a_+ = 0.81$, $M_+ = 1.01$, $B_- = 1.99$, $B_+ = 2.01$, $A_+ = 0.003$, $b_+ = 0.0015$, $L_- = 99$, $D_+ = 1.1$. Every member has $a, M, L > 0$ and $c \ge 1 - b_+^2/L_- = 43999999/44000000 > 0$, so the full quadratic Hamiltonian is positive at nonzero momentum.

Use the common analytic rectangle $0 \le x \le 6$, $|y| \le 6$ and the three nodes

$$(x_1, x_2, x_3) = (8/9, 5/2, 40/7), \qquad \rho = 1/100.$$

The heavy block is invertible throughout the rectangle, since $|y - x| \le 12 < L_-$; the uniform geometric ratio is at most $4/33$. For every member of Θ, the reference lower root obeys $r_- x \le y_0 < a_+ x < x$ and its separation g from the other P01 root is at least B_-. For example, the gap inequality follows from $g^2 = B^2 + 2B(a+1-2r)x + (a-1)^2x^2$ and $r < a < 1$. All three radius-ρ disks lie in the rectangle and strictly in $0 < \operatorname{Re} y < x$.

The following bounds enclose the entire disk around each candidate-dependent y_0, rather than just a fitted pole:

$$\begin{aligned} Y_i &= a_+ x_i + \rho, & T_i &= (1-r_-)x_i + \rho, \\ U_i &= (a_+ - r_-)x_i + M_+ + \rho, & \eta_i &= L_- + (1-a_+)x_i - \rho, \\ \epsilon_i &= \frac{T_i}{L_-\eta_i}\left(A_+^2 T_i + b_+^2 Y_i U_i + 2D_+ A_+ b_+ Y_i\right). \end{aligned} \tag{40}$$

Indeed these respectively bound $|y|$, $|y - x_i|$, $|y - ax_i - M|$, and the distance to the heavy denominator. The exact action identity $R = fQ$ then gives $|R| \le \epsilon_i$. Exact rational evaluation verifies

$$\epsilon_i < \rho(B_- - \rho), \qquad \tau_i := 2\epsilon_i/B_- < \rho, \qquad \epsilon_i \le \tau_i(B_- - \tau_i). \tag{41}$$

The pole theorem therefore gives a unique positive simple full-system root with $|y_{\text{light}} - y_0| \le \tau_i$. These are the physical light roots of this formal quadratic model: positivity gives three positive squared frequencies for $x > 0$, and $\det P_3(x, x) = LD^2x$ precludes crossing $y = x$. Continuation from $x = 0$ leaves exactly one root in $0 < y < x$.

Choose the observed channel to be a source coupled linearly to π, with no external source for H. On $0 < y < x$, $|f| < 1/L$, so $|Abf| < |D|$ follows uniformly from $A_+b_+ < 0.9L_-$. The effective off-diagonal mixing is nonzero, which rules out a light eigenvector with zero π component. The positive Hamiltonian gives positive mode norm and hence positive spectral weight for this nonzero Hermitian source overlap. As a direct check on these disks, $\rho < M_-$ and $y_0 < ax$ imply $(P_{\text{eff}})_{ss} < 0$ at the real light root; the determinant derivative is negative there because this is the lowest of the three positive roots and the eliminated heavy block is negative.

4.3 Nonzero model and observation errors

Take the particular interior point

$$(a, B, M, D, A, b, L) = (4/5, 2, 1, 1, 1/500, 1/1000, 100). \tag{42}$$

It corresponds to $h_0 = 1/1000$, $h_1 = 1/250$ and $S_0 = 1/200000$. The reference lower roots are exactly $(22/45, 3/2, 26/7)$. For each node the reproduction code encloses the unique full-action light root by 112 exact rational bisections on $[0, x_i]$. Denote the resulting endpoints by ℓ_i, u_i. Generate the observations by the exact rule

$$\widehat{y}_i = (\ell_i + u_i)/2 + \nu_i, \qquad (\nu_1, \nu_2, \nu_3) = (1, -1, 1)\, 5 \cdot 10^{-9}.$$

The separately assumed simultaneous measurement bound is $m_i = 10^{-8}$. Exact endpoint inequalities verify $0 < |\widehat{y}_i - y_{\text{light},i}| \le m_i$ and $0 < y_{\text{light},i} - y_i \le \tau_i$ at all nodes. The finite inverse uses $e_i = \tau_i + m_i$, with no probabilistic independence or quadrature assumption. For scale only, the actual model shifts are approximately $(6.31 \cdot 10^{-11}, 3.47 \cdot 10^{-10}, 1.20 \cdot 10^{-9})$.

x_i	upper bound on ϵ_i	upper bound on τ_i	upper bound on e_i
8/9	$6.3481 \cdot 10^{-10}$	$6.3800 \cdot 10^{-10}$	$1.063800 \cdot 10^{-8}$
5/2	$5.18872 \cdot 10^{-9}$	$5.21479 \cdot 10^{-9}$	$1.521479 \cdot 10^{-8}$
40/7	$2.986547 \cdot 10^{-8}$	$3.001555 \cdot 10^{-8}$	$4.001555 \cdot 10^{-8}$

The table rounds bounds upward; the inverse uses their exact rational values. The resulting estimates are

$$\widehat{p} \simeq (0.800000083092, 2.000001426126, 1.000000766069)^T,$$

and certified upper bounds are

$$\alpha < 7.52292628 \cdot 10^{-6}, \quad E_p < 1.696673813 \cdot 10^{-5}, \quad \frac{E_p}{\widehat{B} - E_p} < 8.48343498 \cdot 10^{-6} < 10^{-3}.$$

All admissibility margins in (37) hold; $R_r < 1.272515270 \cdot 10^{-5}$ and $|\widehat{a} - \widehat{r}| > E_p + R_r$, so the certified reference interval also avoids $a = r$. Outward rounded reference intervals are

$$\begin{aligned} B &\in [1.999984459388,\ 2.000018392865], \\ \sqrt{B} &\in [1.414208067926,\ 1.414220065218]. \end{aligned} \tag{43}$$

The target 10^{-3} is a chosen true-relative error requirement on B for this synthetic design, not an empirical accuracy claim.

4.4 A separate certificate for the actual heavy gap

The light inverse estimates the reference B. To infer the lower mixed heavy squared gap Y_H at $x = 0$, use the separate exact quadratic heavy equation with $K = (A - Db)^2$ and $L > B$. The previously derived identity

$$(B - Y_H)(L - cY_H) = KY_H/L$$

gives a uniform one-sided shift bound on the same Θ:

$$0 \le B - Y_H \le \Delta_H := \frac{B_+(A_+ + D_+b_+)^2}{L_-(L_- - B_+)} = \frac{64387}{14225200000000} < 4.526264 \cdot 10^{-9}. \tag{44}$$

This conservative envelope retains the sign uncertainty in D. Combining it with the inverse, rather than identifying Y_H with B, yields

$$\begin{aligned} Y_H &\in [\widehat{B} - E_p - \Delta_H, \widehat{B} + E_p] \subset [1.999984454861,\ 2.000018392865], \\ \sqrt{Y_H} &\in [1.414208066326,\ 1.414220065218]. \end{aligned} \tag{45}$$

All square-root endpoints are rounded outward from directed rational enclosures. At (42), $K = 10^{-6} > 0$, $B - Y_H \simeq 2.040816326 \cdot 10^{-10}$, $Y_H \simeq 1.999999999796$, whereas the additional root is $Y_S \simeq 100.000001010204$. The bare mass-squared parameter is $L = 100$. Thus even this small example has distinct heavy scales. The exact heavy target criterion correctly accepts $B - Y_H \le 10^{-8}$ and rejects $B - Y_H \le 10^{-11}$ at this same point.

The lower heavy neighborhood lies inside the declared $|y| \le 6$ domain, while Y_S does not. Its displayed value is an exact full-model quadratic calculation, not a low-energy prediction of a truncated derivative expansion. No quantum remainder, omitted operator budget, measured coupling prior, or physical cutoff has been supplied by this example.

4.5 Reproduction and scope

The files `INPUTS.json`, `check_joint_application.py` and `exact_tools.py`, retained in the internal research archive, specify the complete experiment. `APPLICATION_RESULTS.json` records exact inputs, root enclosures, inverse bounds, heavy bounds, source hashes and decimal displays. Arithmetic assertions use rational coefficients and directed root intervals only. The standalone polynomial checks also verify the source expansion, background cancellation, all Schur entries, the full determinant, the heavy equation, and a nontrivial exact linear-light tuning. No previous P01, P04 or robustness test is repeated. This application establishes a conditional mathematical action-to-inverse design set, with both errors present; physical use still requires independently justified inputs and the missing quantum and omitted-operator control.

5 Source justification and scope of the classical extension

The historical internal matter proposal has the form

$$\mathcal{L}_{\text{old}} = -\tfrac{1}{2}(\partial\sigma)^2 + K(z)X + a(z)X^2 - U(\sigma,\theta) - \tfrac{1}{2}[(\partial\chi)^2 + m_\chi^2(\sigma,\theta)\chi^2], \tag{46}$$

in signature $(-+++)$, with $X = -(\partial\theta)^2/2$, $z = \sigma/f_\sigma$, $a(z) = 5(z - 2/\sqrt{3})^2/(3\Lambda^4)$ and an explicitly phase-dependent potential. The recovered source and P04 use a constant vacuum with two nonzero light eigenmasses. This differs from P01's shift-symmetric action and rolling solution. The original messages, runtime pressure, global potential and intervening P04/S06 records are bound by paths, locations and hashes in the source audit retained in the internal

research archive. Historical proposals are documentary evidence, not independent physical validation.

The explicit conditional class-A proposal introduces an additional Gaussian scalar of constant mass, with

$$\mathcal{L}_A = -\tfrac{1}{2}(\partial S)^2 - \tfrac{1}{2}M_S^2 S^2 + S J_{\rm old}, \qquad J_{\rm old} = g_0(z - 2/\sqrt{3})X. \tag{47}$$

There are no self-interactions or light-dependent S^2 terms in this defined class. Saturating the displayed X^2 contact fixes $g_0^2/(2M_S^2) = 5/(3\Lambda^4)$, conditional on that ansatz, but does not separate the coupling and mass. The leading source square generates $(z - 2/\sqrt{3})^2 X^2$, not the P01 determinant interaction $C(Z^2 - Y_s Y_t)$. The next derivative term acts on the whole source: for $B = z - 2/\sqrt{3}$, it is proportional to

$$BX\Box(BX) = B^2 X\Box X + 2BX\,\partial z \cdot \partial X + BX^2\Box z. \tag{48}$$

These terms must be retained together and can contribute to the quadratic fluctuation action on a rolling background. They do not identify the leading four-derivative tensor with the determinant tensor.

This restriction extends, in a fixed operator basis, to a pure determinant contact generated solely by healthy Gaussian scalar currents linear in Y_s, Y_t, Z: the leading exchange is a positive semidefinite sum of squares. Vanishing Y_s^2 and Y_t^2 coefficients removes both corresponding source components and therefore their cross term. This restricted statement does not exclude other mediator spins, derivative currents, or additional contacts. The original χ^2 portal also has zero light–light and light–χ Hessian entries at $\chi_0 = 0$. It supplies a separate χ block, not a classical correction to P01's two-field kernel. Quantum or nonzero-state effects require further inputs.

5.1 Meaning of the analytic domain

The selected affine source transfers the original type of Gaussian construction to P01; the local subtraction is a new matching choice. No original frozen-vacuum parameter values are imported. P01's determinant coefficient C remains independent.

The exact classical inverse requires a boundary prescription. An ordinary action uses a symmetric or in-out inverse; retarded response is specified through causal equations or a doubled functional. A freely excited homogeneous heavy solution is additional initial data. On the pole-free analytic symbol domain these inverses have the same rational expression.

The geometric bound in the main calculation concerns the stationary linear fluctuation symbol. Nonlinear products with unrestricted Fourier support are not covered. The eliminated-block singularity $y - x = L$ is not automatically a pole of the coupled system, and small $|y - x|$ does not bound energy for nearly lightlike momenta. No physical interaction cutoff is thereby determined.

5.2 Quantum boundary and resumption condition

The Gaussian determinant at fixed metric is independent of the light background. This removes only light-dependent pure-heavy terms. The full fluctuation determinant also contains the light Schur complement. The heavy/mixed separation and the distinction between a nonlocal eliminated action and its local matching expansion are explicit in Ref. [4], Eqs. (53), (60)–(67); finite matching coefficients also require a scheme, as discussed there in Sec. 1.3. Hence Gaussian elimination does not prove zero mixed or light quantum corrections. P04's local ultraviolet residue supplies no finite matching boundary for this different background.

A quantum-controlled continuation requires a source-normalized renormalized light two-point kernel for the selected theory and state, specified finite matching conditions, and a uniform

bound on remaining contributions on the same parameter set and theorem domain. Cuts crossing a complex pole tube or time-dependent states require a revised domain or theorem. A finite one-loop calculation alone does not bound all higher loops and unknown operators. No such complete majorant is currently available. The present result is therefore conditional and classical; it neither sets the quantum remainder to zero nor opens the existing physical-response gates.

6 Scope of the result and physical continuation

The extension replaces an unspecified classical kernel error by the exact response of a specified extra field. Its bound can be evaluated uniformly before fitting the light data. A successful joint certificate then constrains the parameters of the matched reference action, with a separate certificate for a heavy continuation pole. It does not identify the bare mediator mass from three light measurements while all extension parameters are unknown.

The numerical application is a mathematical construction with declared inputs. Experimental error levels, mediator couplings and masses, and a physical interaction-validity domain have not been measured or independently inferred here. A physical application additionally needs their provenance, source calibration, and a bound on contributions outside the declared classical action, including the appropriate quantum and background corrections. A covariance matrix alone does not define a simultaneous hard error set.

The finite matching subtraction is part of the selected model. Other finite local terms change the reference parameters and their interpretation. A constant heavy-only determinant cannot establish that mixed heavy-light matching terms vanish. A pole bound for the quadratic model also does not prove global nonlinear stability or a high-frequency front speed beyond the effective domain.

The remaining physical decision is therefore specific: identify an extended action and finite matching prescription with an independently justified parameter/domain set, or provide measured matching data that select one. No frozen parameter from the distinct constant-vacuum calculation has been silently transferred. Existing project-wide physics gates are unchanged.

Reproducibility and review. The public papers and editable document sources present the selected action and derivations. The source bindings, verification scripts, explicit design inputs and independent internal review are retained in the internal research archive. Its checks are AI-assisted internal mathematical and computational checks, not external peer review. No experimental data, journal acceptance or comprehensive novelty guarantee is claimed.

Author declarations and licensing

Author funding and competing-interest declarations are given in the main paper. This supplement's original text and accompanying explanatory material are licensed under CC BY 4.0; the author's own supplied code is under the MIT License. Third-party material retains its original terms.

References

[1] Lucas Pinol, Sébastien Renaux-Petel, and Denis Werth. The cosmological flow: A systematic approach to primordial correlators, 2023. URL https://arxiv.org/pdf/2312.06559v1. Version 1, inspected PDF pp. 36–38.

[2] Luc Jaulin and Eric Walter. Set inversion via interval analysis for nonlinear bounded-error estimation. *Automatica*, 29(4):1053–1064, 1993. doi: 10.1016/0005-1098(93)90106-4. URL https://webperso.ensta.fr/Jaulin/paper_automatica93.pdf.

[3] David Bindel and Amanda Hood. Localization theorems for nonlinear eigenvalue problems, 2013. URL https://www.cs.cornell.edu/~bindel/papers/2013-simax.pdf. Author manuscript, 22 pages; Lemma 2.1 and Theorem 4.4 inspected.

[4] Brian Henning, Xiaochuan Lu, and Hitoshi Murayama. One-loop matching and running with covariant derivative expansion, 2016. URL https://arxiv.org/abs/1604.01019v1. Version 1. Directly checked: Secs. 1.1, 1.3, 3.1; Eqs. (3), (50)–(53), (60)–(67).