Finite moment information and sharp logarithmic uncertainty in subtracted Volterra responses

Abstract

Consider a scalar second-order Volterra response with a bounded measurable real feedback and an endpoint finite-part output. We determine its leading uncertainty when a primitive source error has an L^1 budget δ, a fixed L^∞ bound A, and finitely many exact continuous moments including total mass. For every fixed moment family, the worst complex absolute response error is asymptotic to $\frac{1}{2}\delta\log(AT/\delta)$, uniformly over a bounded feedback class on $[0,T]$. Additional fixed moments do not reduce the leading coefficient. An explicit lower construction has smooth compactly supported errors, preserves every source endpoint jet, and lies in a fixed nonnegative uniformly bounded source class. The proof combines a uniform logarithmic source-kernel expansion with two concentrations at different scales and an exact interior moment repair. We also identify the absolute logarithmic source domain and distinguish source truncation from finite-part time regularization. The result is a sharp information bound for this operator class; it is not a new general recovery algorithm or a physical renormalization prescription.

1 Introduction

An error bound for a primitive source does not automatically control a singular response of the solution it drives. Ordinary moments can be very accurate while information near a singular endpoint remains incomplete. This distinction matters when an error certificate consists of an integral residual, a peak bound, and a finite collection of exactly matched source moments.

We study this information contract for

$$y''(t) = p(t)y(t) + f(t), \qquad 0 < t < T, \tag{1}$$

with exact initial traces and real measurable feedback satisfying $\|p\|_\infty \le M$. The output subtracts the initial value and slope from an integral with weight t^{-2}. Transferring this output to the source produces a logarithmic kernel. The main question is quantitative: how much do finitely many exact moments improve the worst response uncertainty of a source error that is small in L^1 but has a fixed peak cap?

For a fixed finite family of real continuous moment kernels containing the constant function, we prove

$$\sup_{\substack{\|e\|_1\le\delta,\ \|e\|_\infty\le A\\ \int \phi_k e=0\ (0\le k<m)}} |\mathcal{L}_p(e)| = \left(\tfrac{1}{2}+o(1)\right)\delta\log(AT/\delta). \tag{2}$$

Here A, T, M and the moment family are fixed, and the remainder is uniform in p. The lower bound remains true using smooth real errors in a nonnegative bounded source class. Thus neither additional fixed continuous moments nor exact source endpoint jets yield a Lipschitz response bound in the L^1 residual alone. The endpoint jets are those of the source, not unspecified terminal data of the solution.

The general tools behind this analysis are established. Endpoint finite-part subtraction is standard; an analytic quadrature treatment is given by Ogata [1]. Operational convolution

calculus also treats initial polynomial terms and finite-part integrals [2, 3]. We do not transfer analytic quadrature assumptions to merely measurable feedback. Weighted Hardy theory provides broader context for the source-integrability question [4]; the particular logarithmic identity needed here is proved directly.

Finite moment bounds and pointwise dual majorants belong to generalized moment duality; see Shapiro [5, Section 3, especially (3.8) and Proposition 3.4]. Optimal recovery with fixed linear observations and intersections of norm constraints is also established; Foucart [6, Section 3.1, Theorem 4] gives a general dual formulation. Our contribution is the explicit leading constant in (2), together with the smooth moment-preserving lower construction and its uniform transfer to (1). We do not claim a new general duality principle. The reviewed comparison results do not evaluate this specific logarithmic asymptotic or supply the joint positive-source and endpoint-jet construction.

2 The source response and its domain

Fix $T > 0$ and $M \geq 0$, and write

$$\mathcal{P}_M = \{p : (0,T) \to \mathrm{R} : \ p \text{ measurable}, \ \|p\|_\infty \leq M\}.$$

For $p \in \mathcal{P}_M$ and $f \in L^1(0,T;\mathrm{C})$, the initial-value problem has a unique $W^{2,1}$ solution. With initial data $y(0) = a$, $y'(0) = b$, our convention is

$$\mathcal{R}_p[y] = \lim_{\varepsilon \downarrow 0} \left\{ \int_\varepsilon^T \frac{y(t)}{t^2}\,\mathrm{d}t - \frac{a}{\varepsilon} + b\log(\varepsilon/T) \right\}, \tag{3}$$

when this limit exists. The logarithmic scale is T, and no further finite counterterm is added. For the homogeneous solution y_h, the limit is

$$\mathcal{R}_p[y_h] = -\frac{a}{T} + \int_0^T \frac{y_h(t) - a - bt}{t^2}\,\mathrm{d}t. \tag{4}$$

The term $-a/T$ is the finite upper boundary contribution. The source solution y_f has zero initial traces; set

$$\mathcal{L}_p(f) = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^T \frac{y_f(t)}{t^2}\,\mathrm{d}t. \tag{5}$$

The full response is the sum of (4) and (5) whenever the latter exists.

Lemma 2.1 (Uniform logarithmic kernel)**.** *For $p \in \mathcal{P}_M$, let G_p be the real Green kernel determined by*

$$G_p(t,s) = t - s + \int_s^t (t-r)p(r)G_p(r,s)\,\mathrm{d}r. \tag{6}$$

For $0 < s < T$ the source kernel satisfies

$$\psi_p(s) := \int_s^T \frac{G_p(t,s)}{t^2}\,\mathrm{d}t = w(s) + r_p(s), \qquad w(s) = \log(T/s) - 1 + s/T, \tag{7}$$

where

$$\|r_p\|_\infty \leq B, \qquad B = \frac{MT^2}{12}\cosh(\sqrt{M}\,T). \tag{8}$$

The formula includes $M = 0$ with $B = 0$.

Proof. The jth iterated term in (6) is bounded by $M^j(t-s)^{2j+1}/(2j+1)!$. Consequently the series and its first time derivative converge uniformly. It solves (6), has the required traces, and satisfies the differential equation in the weak sense. For $u = t - s$,

$$|G_p(t,s) - u| \le \sum_{j\ge 1} \frac{M^j u^{2j+1}}{(2j+1)!} \le \frac{Mu^3}{6}\cosh(\sqrt{M}\,T).$$

Integrating the first term gives w. Since $(t-s)^3/t^2 \le t$, the remainder is bounded by (8). This proof uses only $|p| \le M$, not positivity or differentiability of p. □

Proposition 2.2 (Absolute source domain)**.** *For $f \in L^1(0,T;\mathrm{C})$,*

$$\int_0^T |\psi_p(s)f(s)|\,\mathrm{d}s < \infty \quad\Longleftrightarrow\quad \int_0^T \log(T/s)\,|f(s)|\,\mathrm{d}s < \infty. \tag{9}$$

In this domain, $\mathcal{L}_p(f) = \int_0^T \psi_p f$. For a nonnegative real source, the condition is also necessary and sufficient for a finite cutoff limit in (5); if it fails, the cutoff response tends to $+\infty$.

Proof. The equivalence follows from

$$|\psi_p(s)| \le \log(T/s) + B, \qquad \log(T/s) \le |\psi_p(s)| + 1 + B,$$

and $f \in L^1$. At positive ε, Fubini applies to the Green formula. The contribution of $t - s$ to the cutoff kernel is

$$w_\varepsilon(s) = \begin{cases} \log(T/\varepsilon) - s/\varepsilon + s/T, & s < \varepsilon, \\ w(s), & s \ge \varepsilon. \end{cases}$$

It is nonnegative and increases to w as ε decreases. The feedback remainder is uniformly bounded by B and converges pointwise, so its integral against any L^1 source converges by dominated convergence. Monotone convergence proves the nonnegative assertion, including divergence. Absolute integrability justifies the signed or complex identity. □

Remark 2.3. For a general L^1 source, $y_f(t)/t \to 0$ at the initial endpoint; a uniform quadratic vanishing estimate need not hold. Proposition 2.2 concerns absolute source integration. It does not rule out conditional convergence for selected signed sources and does not identify all such limits with an unqualified Fubini interchange.

3 Sharp uncertainty under fixed moment information

Let $A > 0$. Fix real $\phi_0, \dots, \phi_{m-1} \in C[0,T]$, linearly independent on $(0,T)$, with $\phi_0 = 1$. The same formulation applies when the constant function belongs to the observation span, after a change of basis and removal of dependencies. Define

$$\mathcal{E}_\delta = \left\{ e \in L^\infty(0,T;\mathrm{C}) : \ \|e\|_1 \le \delta,\ \|e\|_\infty \le A,\ \int_0^T \phi_k e = 0 \ \ (0 \le k < m) \right\}, \tag{10}$$

$$U_p(\delta) = \sup_{e\in\mathcal{E}_\delta} |\mathcal{L}_p(e)|\,. \tag{11}$$

Every error in (10) lies in the absolute domain. The family of observations is fixed as $\delta \to 0$.

Theorem 3.1 (Fixed-moment uncertainty)**.** *For fixed A, T, M and the above moment family,*

$$\lim_{\delta\downarrow 0} \sup_{p\in\mathcal{P}_M} \left| \frac{U_p(\delta)}{\delta\log(AT/\delta)} - \frac{1}{2} \right| = 0. \tag{12}$$

For $0 < \delta \le 2AT$, an explicit upper bound is

$$U_p(\delta) \le \frac{\delta}{2} \log \frac{2AT}{\delta} + \frac{\delta^2}{8AT} + B\delta. \tag{13}$$

The same leading lower bound holds if the supremum is restricted to real $e \in C_c^\infty(0,T)$ satisfying $\|e\|_\infty \le A/2$ and $0 \le A/2 + e \le A$. In particular, the lower witnesses preserve all source endpoint jets and every prescribed moment of the constant source $f_0 = A/2$.

3.1 The upper bound

For $e \in \mathcal{E}_\delta$, rotate the output phase and take the real part: $q = \mathrm{Re}(e^{-i\theta}e)$, where $\mathcal{L}_p(q) = |\mathcal{L}_p(e)|$. This uses the reality of both p and the moment kernels. All null moments are preserved, and neither norm increases. Total mass gives $\int q_+ = \int q_- = m_q \le \delta/2$. Since $w \ge 0$,

$$\mathcal{L}_p(q) \le \int_0^T w(s) q_+(s)\, \mathrm{d}s + B\delta.$$

For $0 \le g \le A$ of mass m_q, the decreasing-weight rearrangement bound is

$$\int_0^T wg \le A \int_0^{m_q/A} w(s)\, \mathrm{d}s = m_q \log(AT/m_q) + \frac{m_q^2}{2AT}. \tag{14}$$

One direct proof subtracts the function $A\chi_{(0,m_q/A)}$ and the constant weight $w(m_q/A)$: the resulting product is nonpositive pointwise, and equal mass cancels the constant part. The last expression in (14) increases on $0 \le m_q \le AT$, since its derivative is $w(m_q/A) \ge 0$. This proves (13).

3.2 A fixed interior right inverse

Linear independence supplies distinct interior points whose evaluation matrix has full rank. Choose nonnegative normalized $\theta_j \in C_c^\infty(0,T)$ with sufficiently small disjoint supports around those points. Continuity makes

$$V_{kj} = \int_0^T \phi_k(s)\theta_j(s)\, \mathrm{d}s$$

invertible. Fix these functions once and for all. Let $J = [s_*, s^*] \subset (0,T)$ contain their supports, and set

$$D = \sum_{j,k} |(V^{-1})_{jk}|, \qquad H = \max_j \|\theta_j\|_\infty, \qquad C_J = \log(T/s_*) + B. \tag{15}$$

For any real moment vector d, the correction $v = \sum_j (V^{-1}d)_j \theta_j$ satisfies

$$\int \phi_k v = d_k, \quad \|v\|_1 \le D\,\|d\|_\infty, \quad \|v\|_\infty \le DH\,\|d\|_\infty, \quad |\mathcal{L}_p(v)| \le C_J D\,\|d\|_\infty. \tag{16}$$

All constants are independent of p and δ. Existence of the full-rank point set follows because otherwise all evaluation vectors would span a proper subspace of R^m, giving a nontrivial linear relation among the ϕ_k.

3.3 Two scales and exact moment repair

Fix $\rho \in C_c^\infty(1,2)$, $\rho \geq 0$, $\int \rho = 1$, and let $K = \|\rho\|_\infty$. For small δ, define

$$L = \log(AT/\delta) > 1, \qquad r = T\exp(-\sqrt{L}), \tag{17}$$

$$\omega = \max_k \sup_{0\leq s\leq 2r} |\phi_k(s) - \phi_k(0)|, \qquad m_\delta = \frac{\delta}{2(1+D\omega)}, \qquad \ell = \frac{2m_\delta K}{A}. \tag{18}$$

Continuity gives $\omega \to 0$ and $\ell/r \to 0$. For sufficiently small δ we have

$$2r < T, \qquad 2\ell < r, \qquad r + 2\ell < s_*, \qquad \frac{\delta DH\omega}{1+D\omega} \leq \frac{A}{2}. \tag{19}$$

Define equal-mass pulses

$$u_+(s) = \frac{m_\delta}{\ell}\rho(s/\ell), \qquad u_-(s) = \frac{m_\delta}{\ell}\rho((s-r)/\ell).$$

Their supports lie in $(\ell, 2\ell)$ and $(r+\ell, r+2\ell)$, respectively. They have mass m_δ and peak $A/2$. Both are in $(0, 2r)$, away from J. Their moment defect $d_k = \int \phi_k(u_+ - u_-)$ obeys

$$d_0 = 0, \qquad \|d\|_\infty \leq 2m_\delta\omega. \tag{20}$$

Use (16) and set $e_\delta = u_+ - u_- - v$. Then all moments vanish exactly and

$$\|e_\delta\|_1 \leq 2m_\delta + 2Dm_\delta\omega = \delta.$$

The support regions are disjoint. By (19), $\|v\|_\infty \leq A/2$, hence $\|e_\delta\|_\infty \leq A/2$. These are smooth compactly supported errors. The sources $A/2 + e_\delta$ lie between 0 and A, agree in every prescribed moment with $A/2$, and agree with it in their own neighborhoods of both endpoints.

The logarithmic contributions satisfy, uniformly in p,

$$\begin{aligned}
&\mathcal{L}_p(u_+) \geq m_\delta[\log(T/\ell) - \log 2 - 1 - B] \geq m_\delta[L - \log(2K) - 1 - B],\\
&\mathcal{L}_p(u_-) \leq m_\delta[\log(T/r) + B] = m_\delta[\sqrt{L} + B],\\
&|\mathcal{L}_p(v)| \leq 2C_J Dm_\delta\omega.
\end{aligned}$$

Thus, with $C_0 = \log(2K) + 1 + 2B$,

$$U_p(\delta) \geq \frac{\delta}{2(1+D\omega)}[L - \sqrt{L} - C_0 - 2C_J D\omega]. \tag{21}$$

The expression need not be positive away from the small-δ regime. Dividing by δL, using $\omega \to 0$, and combining with (13) proves Theorem 3.1. No numerical extrapolation is involved.

4 Information and source certificates

4.1 What the sharp rate says

The uncertainty set (10) is centrally symmetric. For a fixed center f_0 in the absolute source domain, on the affine class $f_0 + \mathcal{E}_\delta$, any estimator using only the stated moments receives identical data for $f_0 + e$ and $f_0 - e$. Its largest error on this pair is at least $|\mathcal{L}_p(e)|$. The central estimate $\mathcal{L}_p(f_0)$ attains the radius (11). This is the usual radius-of-information interpretation, not a new recovery principle. The smooth lower witnesses also keep both sources in $[0, A]$ when $f_0 = A/2$.

Extra fixed continuous moments cannot change the leading rate or its coefficient. They affect the nonasymptotic repair constants and continuity modulus. We make no statement uniform in the number of moments, in a moment family changing with δ, or in the conditioning of its right inverse.

If *all* moment conditions are removed, the same argument with one positive pulse gives

$$U_p^{\mathrm{free}}(\delta) = (1 + o(1))\delta \log(AT/\delta), \tag{22}$$

uniformly in p. Deleting only a separately named mass observation is different. For example, retaining $\int(1 + s/T)e = 0$ forces the positive mass of the real projection to be at most $2\delta/3$, and therefore the leading upper coefficient is at most $2/3$. No classification of all observation spaces without the constant function is asserted.

4.2 A primitive endpoint-mass certificate

Let $F_e(r) = \int_0^r |e(s)|\,\mathrm{d}s$. Tonelli yields

$$\int_0^T \log(T/s)|e(s)|\,\mathrm{d}s = \int_0^T \frac{F_e(r)}{r}\,\mathrm{d}r. \tag{23}$$

If a proved primitive envelope gives $F_e(r) \le E(r)$ for $0 < r \le h$, then

$$\left|\int_0^h \psi_p(s)e(s)\,\mathrm{d}s\right| \le E(h)[\log(T/h) + B] + \int_0^h \frac{E(r)}{r}\,\mathrm{d}r. \tag{24}$$

For example, $E(r) = Cr^\alpha$, $\alpha > 0$, gives $Ch^\alpha[\log(T/h) + B + 1/\alpha]$. This is a source-coordinate tail in the full output to T. The different outer-time regularization error is derived in Appendix B. The source uncertainty is paid once in each applicable error relation; the two cutoffs are not interchangeable error accounts.

4.3 Scope of the conclusion

A known common gap $e = 0$ on $(0, s_0)$ changes the situation: directly from Lemma 2.1, $|\mathcal{L}_p(e)| \le [\log(T/s_0) + B]\,\|e\|_1$. The lower witnesses have no such uniform gap. Similarly, measuring the singular output itself is outside the continuous-kernel observation class and removes its uncertainty by definition. The nonnegative example uses a nondegenerate moment fiber; total mass zero would force a nonnegative source to vanish.

The equation, source spaces, peak cap and feedback bound are explicit mathematical assumptions. No primitive physical source values are inferred. The time finite part (3) is not identified with a field-theoretic renormalization rule. The theorem concerns a scalar response and does not establish matrix, momentum, contact-term or ultraviolet identities for another operator. It supplies a sharp source-information requirement and a constructive lower bound, not a numerical speedup or a bit-complexity result.

A Existence, traces and the finite boundary term

Write $(K_p u)(t) = \int_0^t (t - s)p(s)u(s)\,\mathrm{d}s$. For $u \in C[0, T]$, iteration gives

$$|K_p^n u(t)| \le \|u\|_\infty \frac{M^n t^{2n}}{(2n)!}.$$

The integral equation for (1) is

$$y(t) = a + bt + \int_0^t (t - s)f(s)\,\mathrm{d}s + K_p y(t).$$

Its right-hand side seed is continuous for $f \in L^1$. The factorial majorant gives uniform convergence of the resolvent series, hence existence. For the difference of two solutions, iterating the homogeneous equation and letting $n \to \infty$ gives uniqueness. Since $py + f \in L^1$, the solution is in $W^{2,1}$ and satisfies the initial traces. The Green formula follows either from the same ordered-integral construction or from the corresponding first-order system.

For the zero-trace source solution,

$$\frac{|y_f(t)|}{t} \le \cosh(\sqrt{M}\,T) \int_0^t |f(s)|\,\mathrm{d}s \longrightarrow 0,$$

and the first derivative has the analogous bound without division by t. This explains why the trace remains zero even when the stronger estimate $y_f = O(t^2)$ is unavailable. The source domain is then decided by Proposition 2.2, not by an assumed pointwise quadratic bound.

For the homogeneous solution, the resolvent estimate gives, for $0 \le t \le h \le T$,

$$|y_h(t)| \le (|a| + h|b|)\cosh(\sqrt{M}\,h), \qquad \|y_h''\|_{L^\infty(0,h)} \le M(|a| + h|b|)\cosh(\sqrt{M}\,h). \tag{25}$$

Consequently $y_h - a - bt = O(t^2)$. The exact identities

$$\int_\varepsilon^T \frac{a}{t^2}\,\mathrm{d}t = \frac{a}{\varepsilon} - \frac{a}{T}, \qquad \int_\varepsilon^T \frac{b}{t}\,\mathrm{d}t = b\log(T/\varepsilon)$$

prove (4) with the convention (3). Changing the logarithmic scale would add a specified finite multiple of b; no scale-independence assertion is made.

B Source truncation and time regularization

The source-tail estimate (24) follows from the exact identity

$$\int_0^h \log(T/s)|f(s)|\,\mathrm{d}s = F_f(h)\log(T/h) + \int_0^h \frac{F_f(r)}{r}\,\mathrm{d}r, \qquad F_f(r) = \int_0^r |f(s)|\,\mathrm{d}s. \tag{26}$$

It concerns sources located before h, whose solutions may contribute to the entire response time interval up to T.

In contrast, the finite-part time regulator removes the outer interval $0 < t < h$. For the zero-trace source part its exact kernel is

$$\mathcal{L}_p(f) - \int_h^T \frac{y_f(t)}{t^2}\,\mathrm{d}t = \int_0^h \psi_{p,h}(s)f(s)\,\mathrm{d}s, \qquad \psi_{p,h}(s) = \int_s^h \frac{G_p(t,s)}{t^2}\,\mathrm{d}t.$$

For f in the absolute domain, this follows by absolute Fubini. Lemma 2.1 applies with upper time h, so

$$\psi_{p,h}(s) = \log(h/s) - 1 + s/h + r_{p,h}(s), \qquad |r_{p,h}(s)| \le B_h := \frac{Mh^2}{12}\cosh(\sqrt{M}\,h).$$

If $F_f(r) \le E(r)$ for $0 < r \le h$ and $\int_0^h E(r)/r\,\mathrm{d}r < \infty$, the source part of the regulator error obeys

$$\left|\int_0^h \psi_{p,h}(s)f(s)\,\mathrm{d}s\right| \le \int_0^h \frac{E(r)}{r}\,\mathrm{d}r + B_h E(h). \tag{27}$$

There is no $E(h)\log(T/h)$ term here because this kernel ends at h. For the full solution with exact initial traces, (25) additionally gives

$$\left|\mathcal{R}_p[y] - \left\{\int_h^T \frac{y(t)}{t^2}\,\mathrm{d}t - \frac{a}{h} + b\log(h/T)\right\}\right| \le \frac{Mh}{2}(|a| + h|b|)\cosh(\sqrt{M}\,h) + \int_0^h \frac{E(r)}{r}\,\mathrm{d}r + B_h E(h). \tag{28}$$

The first term pays the homogeneous jet-subtracted remainder; the remaining terms pay the source part once. For $M = 0$ the homogeneous and feedback remainder terms vanish exactly, while the source integral remains. Equations (24) and (28) answer different truncation questions and are not cumulative payments for one unspecified cutoff.

C Why a peak or local-mass contract is material

Without a peak cap or comparable local-mass control, even local smooth nonnegative sources with exact finite moment data can have unbounded response. This strengthens the domain description by exhibiting compatible source perturbations; the general unbounded-objective moment mechanism itself is established background [5].

Retain the fixed moment repair (16) and any nonnegative $f_0 \in C^\infty[0,T]$ with a positive reserve on J. Let ρ be as above, $d_n = 1/n$, and $a_n = (T/4)e^{-n^2}$. Add

$$u_n(s) = \frac{d_n}{a_n}\rho(s/a_n).$$

Its moment vector has norm $O(d_n)$ because the moment kernels are bounded. Correct that vector with v_n supported in J. Then

$$\int \phi_k(u_n - v_n) = 0, \qquad \|v_n\|_1 + \|v_n\|_\infty = O(d_n).$$

For sufficiently large n the pulse is disjoint from J, and the positive reserve makes $f_n = f_0 + u_n - v_n$ nonnegative. Every $f_n - f_0$ is smooth and compactly supported in $(0,T)$, every prescribed moment is retained exactly, and $\|f_n - f_0\|_1 \to 0$. In particular, the constant moment preserves the exact total mass of these nonnegative sources. Nevertheless, Lemma 2.1 gives

$$\mathcal{L}_p(f_n) - \mathcal{L}_p(f_0) = d_n \log(T/a_n) + O(d_n) \longrightarrow +\infty,$$

uniformly in $p \in \mathcal{P}_M$. The peak heights are not uniformly bounded in this construction. Theorem 3.1 is the distinct quantitative statement that remains after a fixed peak cap has been paid.

The positive-reserve hypothesis identifies a nondegenerate moment fiber. It is essential to the claimed construction and does not imply that all nonnegative moment fibers admit such perturbations. Likewise, preservation of each source endpoint jet does not impose a common endpoint exclusion interval or a uniform bound on higher derivatives throughout the interval.

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