Massive one-loop pole residues in a rank-one derivative scalar effective theory

Abstract

We calculate the primitive one-loop four-point ultraviolet pole generated by a linear field-dependent kinetic coefficient and a quadratic rank-one four-derivative interaction in a two-scalar effective theory. Unequal internal masses and arbitrary external field labels are retained in an exact off-shell spectral-moment representation. We give the complete identical-field massive projection of the three contributing derivative graph classes. The quintic–cubic bubble has a nonzero off-shell residue but no coefficient of stu, including on the massive identical shell. The quartic bubble contributes both an internal-mass six-momentum pole and an external-mass term descending from its eight-momentum pole. The quartic–cubic–cubic triangle has a positive coefficient multiplying ab^2 in its identical-leg stu projection. Sewing the required three-point ultraviolet vertex to a tree cubic cancels one part of that coefficient; the surviving mixed-derivative term is a field-space curvature contraction. Exact rational output and a standard-library implementation accompany the calculation. The results concern local poles and a declared massive polynomial basis, not finite matching, a complete renormalized amplitude or a mediator-mass determination.

1 Introduction and relation to existing results

Derivative scalar effective theories provide a useful setting in which to distinguish an off-shell counterterm, its mass-shell projection and an on-shell combination of reducible and irreducible diagrams. These distinctions matter when several nonrenormalizable vertices are inserted in a massive loop. Setting the external legs massless too early discards some local terms, while treating a selected one-particle-irreducible (1PI) coefficient as an amplitude coefficient can retain a coordinate-dependent contribution.

We study a factorized two-scalar interaction family and calculate its full primitive four-point pole for the three derivative graph classes that can feed six external momenta. The main outputs are the off-shell flavour tensor, the complete massive identical-leg polynomials in Eqs. (15)–(17), and the distinction between the primitive coefficient (18) and the coefficient after necessary three-point sewing, Eq. (34). A fixed numerical example illustrates that the external-mass descent and the sewing cancellation can both be substantial relative to the corresponding primitive terms. The parametric result, rather than those numerical inputs, is the object of the calculation.

The methods themselves are established. Massive scalar interactions with arbitrarily many derivatives can be treated by Feynman parameters, momentum shifts and angular averages [1]. Scalar EFT renormalization in off-shell bases and nonlinear nonrenormalization results have been studied much more generally [2, 3]. Mass-induced lower-derivative renormalization is also known [4, sec. 5]. We use these methods without proposing a new general ultraviolet algorithm.

The closest explicit massless four-derivative comparison is the multi-field example of Ref. [5, eqs. (17)–(20)]. In its single-field normalization $\widetilde{g} = 2a$, its crossed massless pole agrees with our $(7/10)a^2e_2^2$. This shared limit is not a new result. Our calculation retains unequal internal masses, the quintic–cubic and quartic–cubic–cubic classes generated by the field-dependent

kinetic coefficient, and the off-shell terms required to distinguish massive projection from reducible sewing.

Field-space geometry already organizes scalar amplitudes and matching [5–7]. In particular, cubic two-derivative vertices can be removed at a vacuum by choosing normal coordinates [7, sec. 4]; their individual 1PI contributions cannot therefore be observables. The explicit cancellation below is an application of this known principle. Higher-order field redefinitions and equation-of-motion (EOM) insertions must be tracked consistently [8]. Neither a local pole nor a field-independent heavy-only determinant determines finite heavy–light matching [9].

2 Theory, domain and conventions

Work at a constant stationary vacuum in flat spacetime with signature $(-+++)$. Let σ, θ denote fluctuations before a constant canonical transformation. The relevant local action is

$$\begin{aligned}\mathcal{L} = & -\frac{1}{2}(\partial\sigma)^2 - \frac{1}{2}K(\sigma)(\partial\theta)^2 + \frac{1}{4}a(\sigma)[(\partial\theta)^2]^2 - U(\sigma,\theta) \\ & -\frac{1}{2}[(\partial\chi)^2 + m_\chi^2(\sigma,\theta)\chi^2], \end{aligned} \tag{1}$$

$$K(\sigma) = K_0 + b\sigma, \qquad a(\sigma) = a + a_1\sigma + \tfrac{1}{2}a_2\sigma^2. \tag{2}$$

Here $K_0 > 0$, U is a polynomial of degree at most four, and m_χ^2 is analytic near the vacuum. No derivative portal or independent χ^4 term is added. The original application has a quadratic dependence on the unshifted radial field and a trigonometric phase dependence in m_χ^2; the full Taylor series is allowed. Constant quadratic terms belong to the propagators, and linear terms vanish by stationarity. Unknown additional Wilson operators are outside the specified truncation.

Set

$$\sigma = z_i\phi_i, \qquad \theta = r_i\phi_i, \qquad z\cdot z = 1, \quad z\cdot r = 0, \quad r\cdot r = \rho = K_0^{-1}. \tag{3}$$

The constant transformation also diagonalizes the quadratic mass matrix; the eigenvalue squares are $M_i > 0$. We use $i = 1, 2$ for the original system. The tensor-contraction formulas extend algebraically to a finite diagonal spectrum with the same factorized vertices; this extension does not add fields to the original example. A field used as an asymptotic external particle must be stable in the stated matter sector. An unstable eigenmode can still label an off-shell vertex or an internal line.

All momenta enter their vertex. The propagator is $-i/(q^2 + M_i - i0)$ and a vertex is iD_n, where D_n is real. Write

$$W(k_1,k_2,k_3,k_4) = (k_1\cdot k_2)(k_3\cdot k_4) + (k_1\cdot k_3)(k_2\cdot k_4) + (k_1\cdot k_4)(k_2\cdot k_3). \tag{4}$$

The symmetric derivative vertices are

$$D_3 = b\sum_h z_{i_h}\prod_{j\neq h} r_{i_j}(k_j\cdot k_l), \qquad \{j,l\} = \{1,2,3\}\setminus\{h\}, \tag{5}$$

$$D_4 = 2a\prod_{j=1}^{4} r_{i_j}\, W(k_1,k_2,k_3,k_4), \tag{6}$$

$$D_5 = 2a_1\sum_h z_{i_h}\prod_{j\neq h} r_{i_j} W(k_j : j\neq h). \tag{7}$$

There is also a six-leg, four-derivative vertex proportional to a_2. In dimensional regularization $d = 4 - 2\epsilon$, define the pole polynomial $\mathcal{F}$ by

$$\mathcal{M}^{(4)}_{\text{loop,div}} = \frac{\mathcal{F}}{16\pi^2\epsilon}. \tag{8}$$

The quantities are amputated vertices; no normalization factor for an identical external two-particle channel is included. Those factors belong to a later partial-wave construction [10].

For identical external eigenfield e, put $r = r_e$, $z = z_e$ and $\mu = M_e$. The Mandelstam variables obey $s + t + u = 4\mu$. Our unshifted polynomial basis is

$$e_2 = st + su + tu, \qquad e_3 = stu. \tag{9}$$

Mass-dependent changes of this basis alter the assignment of lower polynomial coefficients; the choice in (9) is part of every coefficient statement below. In particular, the coefficient of e_3 is read while retaining e_2^2 as an independent basis element. It is not the homogeneous cubic part obtained by expanding everything in independent s, t after eliminating u.

3 Closure of the primitive six-momentum classes

For a connected one-loop graph with four external legs,

$$I = V, \qquad \sum_v (n_v - 2) = 4, \qquad \delta = 4 - 2V + \sum_v d_v. \tag{10}$$

With all two-leg masses in the propagator and no one-leg interactions, $n_v \geq 3$. The only valence partitions are (6), $(5,3)$, $(4,4)$, $(4,3,3)$ and $(3,3,3,3)$. A one-loop 1PI graph has no bridge, so its internal graph is respectively a tadpole, bubble, bubble, triangle or box. The maximal degrees are $6, 6, 8, 6, 4$.

The potential supplies only zero-derivative cubic and quartic vertices. A portal term $\phi^k \chi^2$ has valence $k+2$ and zero derivatives. Although the portal series is infinite, (10) restricts it to $k \leq 4$ here. Replacing a derivative vertex in a non-tadpole candidate by an available potential or portal vertex lowers δ below six. There is no six-external-momentum candidate containing a χ line or an external χ field.

The six-leg tadpole needs a stronger statement than superficial power counting. Its one vertex has only four derivatives, and its propagator carries no external momentum. Contracting two undifferentiated legs can produce an internal-mass-squared times a four-momentum polynomial; contracting two differentiated legs can produce a fourth mass power times a two-momentum polynomial. An odd loop numerator vanishes. None supplies six external momenta. This argument keeps the massive tadpole and does not infer ultraviolet absence from a scaleless integral.

Proposition 1. *For the action (1), the primitive additive six-external-momentum four-point pole can arise only from D_5D_3, D_4D_4 and $D_4D_3D_3$. The D_4D_4 class also has an eight-momentum pole whose massive projection must be retained. This is a candidate-class theorem, not a nonvanishing statement for each projected coefficient.*

Counting a mass as one momentum defines a different expansion: the six-leg tadpole can then belong to total order six through $M_i P_4$. Such terms are not the coefficient of e_3 in (9). A formal expansion in two-leg mass insertions reproduces the large-momentum expansion of the massive propagator; it does not introduce another independent interaction class. The distinction from a complete chiral-order-six calculation is essential [11].

4 Exact off-shell tensor construction

We retain $p_4 = -p_1 - p_2 - p_3$ and the six independent Gram entries $G_{ij} = p_i \cdot p_j$, $1 \leq i \leq j \leq 3$. No external square is set to zero during pole extraction. The bubble factor is $1/2$. There are three unordered pair partitions for D_4D_4, four singleton choices for D_5D_3, and six choices of

the pair at the quartic vertex for $D_4D_3D_3$. The triangle has factor one: its labelled external singletons distinguish the cubic vertices. Internal species are summed on every edge.

For denominators $(q+P_a)^2+M_a$, introduce simplex parameters and shift

$$\ell = q + \sum_a x_a P_a, \qquad \Delta = \sum_a x_a(M_a + P_a^2) - \Big(\sum_a x_a P_a\Big)^2, \quad \sum_a x_a = 1. \tag{11}$$

For a translated numerator monomial $(\ell^2)^r \prod_{j=1}^{2h}(\ell\cdot v_j)$, set $k=r+h$ and $n=k+2-V$. If $n<0$, its UV pole vanishes. Otherwise its normalized pole is the simplex integral of

$$\frac{(-1)^n(k+1)!\,\Delta^n}{2^h(h+1)!\,n!}\sum_{\mathcal{P}}\prod_{(i,j)\in\mathcal{P}} v_i\cdot v_j. \tag{12}$$

The sum runs over pairings $\mathcal{P}$ of the $2h$ vectors. Odd tensor ranks vanish. The parameter moments are

$$\int_{x_a\ge 0} \mathrm{d}^V x\,\delta(1-\textstyle\sum x_a)\prod_a x_a^{j_a} = \frac{\prod_a j_a!}{(\sum_a j_a+V-1)!}. \tag{13}$$

Appendix A derives the normalization. Positive masses and nonexceptional Euclidean external momenta first separate the UV pole from the IR. Only the resulting local polynomial is continued to a physical shell. At a primitive simple pole, the $O(\epsilon)$ part of angular factors changes finite terms, not the residue.

The entire flavour sum can be written in spectral moments

$$R_n = \sum_a r_a^2 M_a^n, \quad S_n = \sum_a z_a^2 M_a^n, \quad C_n = \sum_a r_a z_a M_a^n. \tag{14}$$

For each internal edge, replace a power M_a^n by R_n, S_n or C_n according to its two half-edge field factors. External factors remain r_i or z_i. Thus every species sum is explicit, including unequal masses. In the original canonical frame $R_0=\rho$, $S_0=1$, $C_0=0$.

There are one, fifteen and nine stripped kernels for the three classes: the latter counts assign the one radial leg of each cubic or quintic vertex. They have been evaluated exactly, not left as parameter integrals. The supplementary file `KERNELS_EXACT.json` gives these 25 polynomials. The complete crossed off-shell tensor is `FULL_OFFSHELL_TENSOR.json`; its external radial masks and spectral-moment labels determine arbitrary external field components. Appendix B specifies the representation and routing. It is more compact than repeating all field components in expanded form.

5 Complete massive identical-field projection

For four identical external fields, define $\mathcal{F}_{44}$, $\mathcal{F}_{53}$ and $\mathcal{F}_{433}$ as the three contributions to (8). Their complete mass-shell polynomials, including lower momentum degrees of these graph

classes, are

$$\begin{aligned}\mathcal{F}_{44} = 2a^2r^4\Big\{&\rho^2\Big(16\mu^4 - \frac{16}{3}\mu^2 e_2 + \frac{7}{20}e_2^2 + \frac{16}{15}\mu e_3\Big)\\ &+ \rho R_1\Big(-16\mu^3 + 4\mu e_2 - \frac{5}{2}e_3\Big)\\ &+ \Big(\rho R_2 + \frac{1}{2}R_1^2\Big)\Big(15\mu^2 - \frac{5}{2}e_2\Big)\Big\},\end{aligned} \tag{15}$$

$$\begin{aligned}\mathcal{F}_{53} = a_1 b\Big\{&r^4(\mu\rho + \rho S_1 + R_1)(6\mu^2 - e_2)\\ &+ r^3 z\rho C_1\Big(28\mu^2 - \frac{10}{3}e_2\Big)\\ &+ r^2z^2\Big[\rho^2\Big(2\mu^3 - \frac{1}{6}\mu e_2\Big) + \rho R_1\Big(-10\mu^2 + \frac{1}{3}e_2\Big)\\ &\qquad + \mu(12\rho R_2 + 6R_1^2)\Big]\Big\},\end{aligned} \tag{16}$$

$$\begin{aligned}\mathcal{F}_{433} = 2ab^2\Big\{&r^4\Big[\rho^2\Big(\frac{7}{4}\mu^3 - \frac{2}{3}\mu e_2 + \frac{5}{16}e_3\Big)\\ &\qquad + (\rho^2 S_1 + 2\rho R_1)\Big(-3\mu^2 + \frac{1}{2}e_2\Big)\Big]\\ &+ r^3 z\rho^2 C_1\Big(-10\mu^2 + \frac{1}{3}e_2\Big)\\ &+ r^2z^2\Big[\rho^3\Big(\frac{35}{4}\mu^3 - 2\mu e_2 + \frac{5}{16}e_3\Big)\\ &\qquad + \rho^2 R_1\Big(-15\mu^2 + \frac{9}{2}e_2\Big) - 9\mu(\rho^2 R_2 + \rho R_1^2)\Big]\Big\}.\end{aligned} \tag{17}$$

These are not the entire lower-derivative counterterm action: potential and portal graphs, excluded from the six-momentum classification, can contribute at those lower degrees. The displayed lower terms are the complete projections of the three specified derivative graph classes only.

Every spectral monomial was separately reconstructed as a polynomial in μ, e_2, e_3 using exact rational arithmetic. Thus crossing is checked at the coefficient level, not inferred from a few scattering geometries. The off-shell tensor retains the terms removed or converted by the mass-shell conditions.

Theorem 2. *For the action and conventions of Section 2, the coefficient of $e_3 = stu$ in the primitive derivative block is*

$$\begin{aligned}\kappa_{53} &= 0,\\ \kappa_{44} &= a^2r^4\left(\frac{32}{15}\mu\rho^2 - 5\rho R_1\right),\\ \kappa_{433} &= \frac{5}{8}ab^2\left(r^4\rho^2 + r^2z^2\rho^3\right),\end{aligned} \tag{18}$$

where $\mathcal{F} \supset \kappa e_3$. The positive numerical prefactor $5/8$ in the last expression refers to the real loop-amplitude convention (8); a itself is not assumed positive. The counterterm that subtracts a given amplitude pole has the opposite sign.

5.1 The quintic–cubic zero is an on-shell statement

There is a direct structural explanation for the first line of (18). At the cubic vertex let the singleton momentum be p and the internal momenta be q, r, with $p + q + r = 0$. For fixed species write its numerator as

$$A(q\cdot r) + B(p\cdot r) + C(p\cdot q) = up^2 + vq^2 + wr^2, \tag{19}$$

where $u = (A - B - C)/2$, $v = (-A + B - C)/2$ and $w = (-A - B + C)/2$. With $D_a = q^2 + M_a$ and $D_b = r^2 + M_b$, this is

$$vD_a + wD_b + (up^2 - vM_a - wM_b). \tag{20}$$

The first two terms reduce the bubble to tadpoles with a numerator of external degree at most four. The internal-mass terms also have degree at most four. Only up^2 can multiply a four-momentum bubble polynomial to make six momenta. Hence the highest off-shell term is in the ideal generated by the singleton p^2, separately for each species and partition. It vanishes on a massless shell, while on a massive shell it becomes a mass times a polynomial of degree at most two in Mandelstam variables. This proves the absence of e_3, not the absence of the off-shell residue or of the lower terms in (16).

5.2 The two mass origins in the quartic bubble

The first and second terms of κ_{44} have different off-shell origins. The term $-5a^2r^4\rho R_1$ comes from an explicit internal-mass factor multiplying a six-momentum polynomial. The term $(32/15)a^2r^4\mu\rho^2$ comes from the eight-momentum polynomial under the external shell. In a symmetric polynomial of degree four, the structure $(s + t + u)stu$ becomes $4\mu stu$; off-shell external squares can contribute in the same way. Setting every external leg massless before retaining that polynomial removes this term.

This effect is not an additional diagram class or a new general mass-renormalization mechanism. It is an explicitly calculated contribution in the selected basis. In the simultaneous internal/external massless limit, (15) reduces to

$$\mathcal{F}_{44} = \frac{7}{10}a^2r^4\rho^2e_2^2, \tag{21}$$

in agreement with the massless geometric result discussed in the introduction [5].

5.3 Separate checks

Three checks use information beyond merely re-evaluating the same Feynman-parameter code. First, the inverse-propagator argument (20) predicts a zero on the singleton EOM surface; all fifteen quintic–cubic leading kernels satisfy it. Second, a two-particle cut gives the quartic-bubble mass-independent pole projected onto equal massive external squares. For a single rank-one coupling g, its s-channel residue is

$$\kappa_{44,s}^{P_8} = \frac{g^2s^2}{240}[83s^2 - 264\mu s + 208\mu^2 + (t - u)^2]. \tag{22}$$

Crossing yields $g^2[7e_2^2/10 + 32\mu e_3/15 - 32\mu^2e_2/3 + 32\mu^4]$. Here $g^2 = a^2r^4\rho^2$ for the full rank-one species sum. A direct large-loop-momentum expansion at rational nonexceptional momenta separately checks internal-mass terms, including their linear and quadratic dependence. Third, the orthogonal-direction triangle cut gives $+5ab^2stu/8$ for $r = (1, 0)$, $z = (0, 1)$ and angular external legs. These are targeted independent internal checks; they do not constitute external peer review or a second exhaustive computation of every off-shell coefficient.

6 Three-point sewing and the curvature term

The primitive triangle coefficient contains a term proportional to $r^4\rho^2$ even for an external state in the angular direction, $z = 0$. A connected four-point calculation at the same derivative order also contains the three-point D_4D_3 ultraviolet bubble joined to a tree D_3. We determine its local Mandelstam-degree-three contribution before interpreting the primitive coefficient.

6.1 Which reducible terms can enter

For a three-point one-loop graph, $\sum(n_v - 2) = 3$. Only the D_4D_3 bubble can have degree six. Potential replacements and the D_3^3 triangle have degree at most four; a nonzero D_5 tadpole has at most two external derivatives. Sewing D_4D_3 to D_3 adds two derivatives and one bridge denominator, preserving degree six. Sewing to a potential cubic has degree at most four.

A two-point D_3^2 loop has degree at most four and remains so between two tree cubics with two propagators. A D_4 two-point tadpole has at most two external derivatives. On-shell external residue factors and mass or field shifts depend on fixed masses and multiply tree contacts with Mandelstam degree at most two. Vacuum shifts likewise change only such contact coefficients. They cannot add a degree-three polynomial. These exclusions concern the stated lower vertices; an independent higher-derivative coupling would require its own analysis.

For a sewn UV term, polynomial division by its bridge denominator separates the local polynomial from its propagator-pole remainder. Terms proportional to an internal mass in the three-point residue, or the difference between $1/(P^2 + M)$ and $1/P^2$, lower the leading Mandelstam degree. Thus the desired degree-three sewing coefficient is obtained from the highest homogeneous three-point residue. This is a statement about its local part, not permission to discard the lower-degree or nonlocal counterterm contributions in a complete amplitude.

6.2 Explicit cancellation

Take two momenta k_1, k_2 at the quartic bubble vertex and the singleton $-P$ at the cubic, where $P = k_1 + k_2$. At highest degree the internal species sum has no mass factors. Orthogonality $C_0 = 0$ removes every radial assignment to an internal cubic leg. Only a radial singleton survives, with factor $\rho^2 r_i r_j z_x$.

The stripped numerator contains

$$W(k_1, k_2, q, -q - P)\,[-q \cdot (q + P)]. \tag{23}$$

Since $-q \cdot (q + P) = -[q^2 + (q + P)^2]/2 + P^2/2$, the mass-independent leading residue is $P^2/2$ times the W bubble. At $k_1^2 = k_2^2 = 0$, its normalized residue is $5(P^2)^2/12$. Including the bubble factor gives the three-point pole

$$\Gamma_{3,\mathrm{div}} = \frac{ab\rho^2 r_i r_j z_x}{16\pi^2\epsilon}\frac{5}{24}(P^2)^3 + \text{lower-degree terms}. \tag{24}$$

The coefficient ab is $(2a)b/2$, with no additional vertex ordering factor.

The leading tree cubic with two external e legs is

$$D_{3,eex} = bP^2\left(\frac{1}{2}z_x r^2 - r_x z r\right) + \text{fixed-mass terms}. \tag{25}$$

The real sewn amplitude contains $+\Gamma_3 D_3/(P^2 + M_x)$. The internal contraction is

$$\sum_x z_x\left(\frac{1}{2}z_x r^2 - r_x z r\right) = \frac{1}{2}r^2. \tag{26}$$

There are six choices of the external pair at the quartic vertex, twice each channel. With $P^2 = -s$ and its crossed versions, the leading local contribution is

$$\kappa_{\mathrm{sew}} = -\frac{5}{8}ab^2 r^4 \rho^2. \tag{27}$$

Attaching the bridge at a quartic leg instead leaves the cubic singleton on its true external mass shell. Its leading factor in (20) is then a fixed external mass; the twelve such attachments have at most Mandelstam degree two. They do not add another e_3 term.

Combining (27) with (18), the ab^2 coefficient becomes

$$\kappa_{ab^2}^{\text{connected},\,e_3} = \frac{5}{8} ab^2 r^2 z^2 \rho^3. \tag{28}$$

In particular it vanishes when $z = 0$, although the primitive triangle pole does not.

6.3 Tree-amplitude and geometric check

The cancellation has a direct on-shell check. For the mass-independent leading coefficient define

$$J_{ij,a} = z_a r_i r_j - r_a (z_i r_j + z_j r_i). \tag{29}$$

The complete two-derivative tree exchange is

$$T_{2,ij;kl} = -\frac{b^2}{4} \sum_a \left[s J_{ij,a} J_{kl,a} + t J_{ik,a} J_{jl,a} + u J_{il,a} J_{jk,a} \right]. \tag{30}$$

Projection of the intermediate legs along r gives

$$\sum_{kl} r_k r_l T_{2,ee;kl} = \frac{b^2}{4} z^2 \rho^3 s. \tag{31}$$

The quartic tree's angle-average is $(5a/6) r^2 r_k r_l s^2$. The cut interference has two placements and the identical-intermediate-state factor $1/2$. Its s-channel UV/log coefficient is therefore $5ab^2 r^2 z^2 \rho^3 s^3/24$; crossing gives (28). The on-shell cubic factors cancel propagators in this highest homogeneous coefficient, so only nonexceptional bubble logs and mass-dressed tadpoles enter the UV argument. It does not equate an arbitrary finite absorptive part with a matching constant.

Choose the Riemann convention in which $\mathrm{d}\sigma^2 + K(\sigma)\mathrm{d}\theta^2$ has positive Gaussian curvature for linear nonconstant positive K. At the vacuum,

$$\mathcal{R}_{ijkl} = \frac{b^2 \rho}{4} (z_i r_j - r_i z_j)(z_k r_l - r_k z_l). \tag{32}$$

Then $\mathcal{R}(e, r, e, r) = b^2 z^2 \rho^3/4$, and

$$\kappa_{ab^2}^{\text{connected},\,e_3} = \frac{5a}{2} r^2 \mathcal{R}(e, r, e, r). \tag{33}$$

This identifies the result within the established field-space description of scalar amplitudes [5, 7]. It does not make the intermediate primitive tensor coordinate independent.

The coefficient of $e_3 = stu$ in the stated basis, after this required sewing, is

$$\boxed{\kappa_{\text{local},e_3} = a^2 r^4 \left(\frac{32}{15} \mu \rho^2 - 5 \rho R_1 \right) + \frac{5}{8} ab^2 r^2 z^2 \rho^3.} \tag{34}$$

It remains a coefficient in the declared massive polynomial basis. In a renormalized calculation the corresponding lower-point counterterms must be sewn consistently. We do not identify (34) with a complete physical beta function or a finite observable.

Table 1: Fixed input for the coefficient comparison. The printed precision is not improved by high-precision arithmetic.

Quantity	Value
a	$3.176406919961296 \times 10^{11}\, M_*^{-4}$
b	$3.430519473558199\, M_*^{-1}$
(r_1, r_2)	$(1.0535251834922142, -0.000505830760925431)$
(z_1, z_2)	$(0.000480131572123461, 0.999999884736830)$
(M_1, M_2)	$(2.90211845795 \times 10^{-10},\ 2.30400046423 \times 10^{-9}) M_*^2$

7 A fixed massive two-field example

Table 1 gives one fixed parameter insertion, in a reference mass unit M_*. These are supplied input values, not fitted data or a newly solved vacuum. The coefficient of e_3 is independent of a_1, a_2, so their values are unnecessary for this table. No scattering-energy range, physical cutoff or mediator scale is inferred from the size of the coefficients. Take the lighter eigenfield as the external state. Table 2 separates terms that would be merged in a single number. The external-mass descent is about 0.426666 of the magnitude of the internal-mass quartic-bubble coefficient. The primitive triangle is dominated by the part removed by sewing; its remaining contribution is small because z_1^2 is small. These are two different effects, and neither is captured by quoting only the massless primitive *stu* coefficient.

Table 2: Local e_3 coefficients at the fixed input, in M_*^{-6}. The two totals refer to different diagram sets.

Contribution	Coefficient
D_4D_4: internal-mass P_6	-2.22187×10^{14}
D_4D_4: external-mass descent from P_8	$+9.47994 \times 10^{13}$
$D_4D_3D_3$: primitive triangle	$+3.54565 \times 10^{12}$
Primitive three-class sum	-1.23841×10^{14}
Required degree-three sewing	-3.54565×10^{12}
Remaining ab^2 coefficient	$+8.17365 \times 10^{5}$
Local coefficient including sewing	-1.27387×10^{14}

8 Interpretation and remaining renormalization data

The complete primitive tensor, its identical-shell projection, and the local coefficient after sewing are different mathematical objects. Equations (15)–(17) give the second directly from the first. Equation (34) adds only the further diagrams relevant to the declared local degree-three coefficient. It is not the complete massive amplitude: lower-degree counterterms, propagator-pole pieces, finite functions and renormalization conditions still matter for that amplitude.

There are also two order conventions. At fixed masses, *stu* has Mandelstam degree three. If masses count as momenta, the quartic-bubble structure μstu is total order p^8, while a tadpole term M_iP_4 has total order p^6. Our classification concerns external momentum degree and a specified massive polynomial basis, not the full chiral-order-six problem. This is why both the off-shell degree and the mass origin are preserved in the supplied data.

The local poles determine subtraction requirements in a chosen operator and field basis. They do not fix finite local coefficients. In particular, a proposed extra Gaussian mediator

coupled linearly to a derivative source yields a familiar tree ratio of coupling squared to mass squared [12, 13]; a light-sector pole alone does not separate those parameters. A constant pure-heavy determinant also does not remove mixed heavy–light contributions [9]. No unique ultraviolet completion or mediator mass follows from the calculation here.

The exact output can be used to construct a counterterm basis including redundant operators, or to translate to another basis with all induced terms retained [8]. The needed follow-on object is a specified renormalized amplitude or matching prescription, not a further rounding of the same primitive coefficient. The present result does not substitute an absorptive cut for an independent finite matching condition.

9 Conclusions

We have evaluated the full off-shell primitive three-class pole tensor of the stated derivative scalar family, retaining unequal internal masses and arbitrary external field labels. Its massive identical-field projection is given in closed form. The quintic–cubic bubble provides a concrete nonzero off-shell residue with a vanishing *stu* projection. The quartic bubble separates an internal-mass six-momentum term from an external-mass contribution inherited from eight momenta. The primitive triangle coefficient contains a contribution cancelled by the required three-point sewing; the remaining mixed-derivative term is a curvature contraction.

The contribution is this explicit parametric tensor and the demonstrated relation between its three levels of projection and sewing. The underlying ultraviolet, field-redefinition and geometric methods, as well as the massless quartic-bubble limit, are established. The calculation supplies local renormalization data for the declared theory and conventions. Finite matching, the complete massive renormalized amplitude and mediator identification remain separate problems.

Reproducibility. Exact rational kernels, flavour tensors, projected coefficients, fixed inputs and a Python-standard-library implementation are provided with the manuscript. The implementation uses no symbolic-algebra package, fitted kinematics, numerical integration or parameter scan. The accompanying record distinguishes checks of mathematical identities, targeted independent internal checks and computational resource measurements. These checks are not external scientific peer review.

A Normalization of the local ultraviolet functional

For a scalar radial numerator, the continued Euclidean integral is

$$\int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{(\ell^2)^k}{(\ell^2+\Delta)^V} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(k+d/2)\Gamma(V-k-d/2)}{\Gamma(d/2)\Gamma(V)} \Delta^{k+d/2-V}. \tag{35}$$

At $d = 4-2\epsilon$ and $n = k+2-V \geq 0$, the second gamma function has residue $(-1)^n/n!$. A rank-$2h$ angular contraction contributes the sum of pairings divided by $d(d+2)\cdots(d+2h-2)$. At four dimensions this denominator is $2^h(h+1)!$. Multiplying by the Feynman-parameter factor $\Gamma(V)$ gives (12). The elementary scalar bubble consequently has normalized pole $+1$, the tadpole has $-M$, and the bubble with numerator ℓ^2 has $-2M$ for equal masses. These checks fix both normalization and the massive sign convention.

For the one-loop graphs considered here $I = V$. Their vertex/propagator phase is $i^V(-i)^I = 1$; the mostly-plus Wick-rotated measure supplies the remaining i in $i\mathcal{M}$. Thus the real residue has the Euclidean polynomial sign used above. A scalar quartic interaction gives the familiar three bubble contribution $3\lambda^2/2$ in the normalization (8). There is no extra triangle phase that reverses the sign of (18).

B Graph kernels and the complete off-shell tensor

The following representatives specify all momenta and internal mass labels. Each listed vertex has vanishing total incoming momentum.

$$\begin{aligned} D_4D_4: \quad & P = p_1 + p_2, \\ & v_A = (p_1, p_2, q, -q - P), \quad v_B = (p_3, p_4, -q, q + P), \\ & \text{denominators: } q^2 + M_A,\ (q + P)^2 + M_B; \end{aligned} \tag{36}$$

$$\begin{aligned} D_5D_3: \quad & P = p_1 + p_2 + p_3, \\ & v_A = (p_1, p_2, p_3, q, -q - P), \quad v_B = (p_4, -q, q + P), \\ & \text{denominators: } q^2 + M_A,\ (q + P)^2 + M_B; \end{aligned} \tag{37}$$

$$\begin{aligned} D_4D_3D_3: \quad & P = p_1 + p_2, \\ & v_A = (p_1, p_2, q, -q - P), \\ & v_B = (p_3, q + P, -q - P - p_3), \quad v_C = (p_4, q + P + p_3, -q), \\ & \text{denominators: } q^2 + M_A,\ (q + P)^2 + M_B,\ (q + P + p_3)^2 + M_C. \end{aligned} \tag{38}$$

For (36) the stripped numerator is $W(v_A)W(v_B)$. For (37), choose one radial leg in the five-leg vertex and one in the cubic; omit those legs when forming W and the remaining dot product. For (38), multiply $W(v_A)$ by the two cubic dot products for their selected radial legs. This gives $1 + 5 \cdot 3 + 3 \cdot 3 = 25$ kernels.

An off-shell tensor term in the data file consists of an external radial mask, a product of spectral moments, and a rational Gram polynomial. Bit j of the mask selects z_{i_j}; an unset bit selects r_{i_j}. Every mass power on an internal edge is replaced by the moment with the same half-edge types, as in (14). Multiply the class sum by

$$2a^2 \quad (D_4D_4), \qquad a_1 b \quad (D_5D_3), \qquad 2ab^2 \quad (D_4D_3D_3). \tag{39}$$

These factors already include each graph's symmetry factor; the file already includes all external assignments. This is a fully evaluated finite representation of every four-point field component, with no implicit momentum integral or unspecified species sum.

The generic representation retains C_0, S_0; the original canonical representation sets $C_0 = 0, S_0 = 1$. The former is useful for checking tensor contractions, but does not change the original action. The off-shell output contains 119, 492 and 551 nonzero Gram-monomial entries, respectively, in the generic spectral representation. Counts describe a data representation, not independent physical operators or a measure of scientific validity.

C Projection and reproducible data contract

For identical external mass square μ, the exact Gram substitution is

$$G_{11} = G_{22} = G_{33} = -\mu, \quad G_{12} = \mu - \frac{s}{2}, \quad G_{13} = \mu - \frac{t}{2}, \quad G_{23} = -\mu + \frac{s + t}{2}. \tag{40}$$

It enforces $p_4^2 = -\mu$ and $u = 4\mu - s - t$. For each spectral monomial and external radial power, the resulting polynomial is exactly reconstructed from $\mu^k e_2^i e_3^j$ with the appropriate homogeneous degree. The difference must vanish as a polynomial; no sampling tolerance is used.

The main machine-readable outputs are:

File	Content
KERNELS_EXACT.json	25 rational routed kernels
FULL_OFFSHELL_TENSOR.json	crossed flavour tensor
MASSIVE_IDENTICAL_PROJECTION.json	complete projected coefficients
SEWING_RESULT.json	three-point check and local sewing
FROZEN_INSERTION.json	fixed numerical input

The accompanying reproduction guide gives commands and expected file hashes. The programs separate the UV functional, graph construction, species contraction and mass-shell reduction. Recorded process limits and measurements distinguish scientific child CPU from document building and other work. Targeted independent checks use an inverse-propagator proof, cut/angle methods and a direct large-radius expansion instead of importing the tensor engine.

References

[1] V. Bhansali and H. Georgi. Running nonlocal lagrangians, 1992. URL https://arxiv.org/abs/hep-ph/9205242v1.

[2] Weiguang Cao, Franz Herzog, Tom Melia, and Jasper Roosmale Nepveu. Renormalization and non-renormalization of scalar EFTs at higher orders, 2021. URL https://arxiv.org/abs/2105.12742v2.

[3] Weiguang Cao, Franz Herzog, Tom Melia, and Jasper Roosmale Nepveu. Non-linear non-renormalization theorems, 2023. URL https://arxiv.org/abs/2303.07391v1.

[4] Garrett Goon, Kurt Hinterbichler, Austin Joyce, and Mark Trodden. Aspects of Galileon non-renormalization, 2016. URL https://arxiv.org/abs/1606.02295v2.

[5] Patrick Aigner, Luigi Bellafronte, Emanuele Gendy, Dominik Haslehner, and Andreas Weiler. Renormalising the field-space geometry. *JHEP*, 07:167, 2025. doi: 10.1007/JHEP07(2025)167. URL https://arxiv.org/abs/2503.09785v3.

[6] Xu-Xiang Li, Xiaochuan Lu, and Zhengkang Zhang. The geometric universal one-loop effective action, 2024. URL https://arxiv.org/abs/2411.04173v1.

[7] Timothy Cohen, Ipak Fadakar, Andreas Helset, and Filippo Nardi. Geometry of soft scalars at one loop, 2025. URL https://arxiv.org/abs/2504.12371v1.

[8] Juan Carlos Criado and Manuel Pérez-Victoria. Field redefinitions in effective theories at higher orders. *JHEP*, 03:038, 2019. doi: 10.1007/JHEP03(2019)038. URL https://arxiv.org/abs/1811.09413v2.

[9] Brian Henning, Xiaochuan Lu, and Hitoshi Murayama. One-loop matching and running with covariant derivative expansion, 2016. URL https://arxiv.org/abs/1604.01019v1.

[10] Mark D. Goodsell and Florian Staub. Unitarity constraints on general scalar couplings with SARAH, 2018. URL https://arxiv.org/abs/1805.07306v1.

[11] J. Bijnens, G. Colangelo, and G. Ecker. Renormalization of chiral perturbation theory to order p^6. *Annals of Physics*, 280:100–139, 2000. doi: 10.1006/aphy.1999.5982. URL https://arxiv.org/abs/hep-ph/9907333v1.

[12] Brian Henning, Xiaochuan Lu, and Hitoshi Murayama. How to use the Standard Model effective field theory, 2014. URL https://arxiv.org/abs/1412.1837v2.

[13] Allan Adams, Nima Arkani-Hamed, Sergei Dubovsky, Alberto Nicolis, and Riccardo Rattazzi. Causality, analyticity and an IR obstruction to UV completion, 2006. URL https://arxiv.org/abs/hep-th/0602178v2.