What ideal rate histories determine in a quadratic critical-reconversion model

Technical note, 23 September 2026

Abstract

We examine a canonical quadratic reconversion equation with an unknown positive rate coefficient, an unknown constant normalization, and a prescribed subcritical state domain. The information experiment grants exactly calibrated input and conversion-rate histories in one model clock; this is stronger than the physical observation model currently supplied by the reconversion proposal. An integral identity characterizes every compatible record and all its parameter realizations. A nonstationary compatible record determines the kinetic ratio and the complete bookkeeping inventory, including its initially unknown value. It does not determine capacity. Conditional on strict survival over a compact past interval, the possible critical conversion rates form exactly the open half-line above the largest past output. This yields sharp future-domain decisions: a predicted output peak not exceeding that past maximum is safe for every compatible realization, whereas a larger finite peak admits both contact and survival. We identify which independent calibrations remove the ambiguity and treat stationary and zero-output exceptions. These are exact model-specific applications of established reservoir inversion and identifiability methods; they do not establish an observed conversion signal or gravitational stability.

1 Question and relation to existing methods

An internal self-regulation law does not by itself specify what a measured history determines about a critical boundary. The phenomenological reconversion proposal considered here uses a dimensionless coordinate x, a positive zone normalization Z, and a positive coefficient G [6]. Its admitted regulated domain is $0 \le x < 1$. We ask a deliberately conditional question: if both rates entering this model were ideally known, would its critical capacity or future domain exit be determined?

The mathematical method is established. Kirchner's storage-discharge framework combines a balance $\dot{S} = P - E - Q$ with a monotone output law $Q = f(S)$ [1, eqs. (1)–(8)]. Identifying $S = q$, $Q = R$, $P - E = F$ and choosing $R = b^2 q^2$ gives $g(R) = \mathrm{d}R/\mathrm{d}q = 2b\sqrt{R}$. The integral relation used below is its quadratic specialization. The absolute origin of our bookkeeping inventory is fixed by that specified output law; it is not a general recovery of the absolute storage origin in an arbitrary reservoir.

Parameter-state transformations that preserve outputs are standard nonidentifiability witnesses. A scaling witness alone need not establish the complete class of indistinguishable realizations [2]. We therefore give a constructive converse for this particular scalar equation. We claim no new general inversion or symmetry method. The purpose of the exact characterization is to separate a reconstructed inventory from an unknown critical capacity, and to state precisely what the entire previously admitted history permits one to infer about a later boundary event. The elementary Riccati solution, square-root equilibrium and linear relaxation rate are also not new results.

Mapping data-compatible parameters into possible model properties is likewise established prediction methodology [3, definitions 4–5, eqs. (8)–(9)]. The results below perform that step exactly for a hidden upper level and a continuous noiseless record, retaining the strict domain and its endpoints. They are not statistical confidence bounds. Analytical contact calculations for quadratic reservoirs also have existing forward applications [4, sec. 2.3]; here the level itself is unknown.

2 Model and information assumptions

On a fixed compact interval $I = [0, T]$, $T > 0$, consider

$$\dot{x} = \frac{F}{Z} - Gx^2, \qquad R = GZx^2, \qquad Z, G > 0. \tag{1}$$

The unknown Z and G are constant. The input $F \geq 0$ is measurable and essentially bounded; the output $R \geq 0$ is continuous. Both are exact rates in a stipulated common mass convention and model clock. A realization has an absolutely continuous state $x \geq 0$ satisfying the differential equation almost everywhere and the continuous output equation everywhere. A *surviving realization* additionally satisfies $x(t) < 1$ for every $t \in I$.

The rates in this information experiment are not asserted to be measured astronomical quantities. Likewise, survival is an additional domain label: rate data alone do not certify that a real object belongs to the prescribed subcritical domain. The coordinate endpoint $x = 1$ is stipulated by the proposal; the scalar equation does not derive a trapped surface, radial instability or collapse at that point.

Define

$$q = Zx, \quad b = \sqrt{G/Z}, \quad \Theta = GZ, \quad Y = \sqrt{R}, \quad B(t) = \int_0^t [F(s) - R(s)]\,\mathrm{d}s. \tag{2}$$

Here q is a derived mass-dimensional bookkeeping variable. It is not thereby identified with a physical mass component. The positive parameters (b, Θ) determine the original ones by

$$Z = \frac{\sqrt{\Theta}}{b}, \qquad G = b\sqrt{\Theta}. \tag{3}$$

The dimensions are $[b] = \mathrm{mass}^{-1/2}\mathrm{time}^{-1/2}$ and $[\Theta] = \mathrm{mass/time}$. The stationary critical supply is Θ, since for constant positive F the mathematical root has $x_*^2 = F/\Theta$. This root remains regular when it crosses $x = 1$; no scalar bifurcation is implied.

3 The complete compatible class

Let

$$\mathcal{K}(F, R) = \{b > 0 : Y(t) = Y(0) + bB(t) \text{ for every } t \in I\}, \qquad R_{\mathrm{p}} = \max_{t \in I} R(t). \tag{4}$$

Theorem 1 (Compatibility and exhaustive realization)**.** *A nonnegative realization of* (1) *exists if and only if* $\mathcal{K}(F, R)$ *is nonempty. All surviving realizations are exactly*

$$b \in \mathcal{K}(F, R), \quad \Theta > R_{\mathrm{p}}, \qquad (Z, G, x(t)) = \left(\frac{\sqrt{\Theta}}{b}, b\sqrt{\Theta}, \frac{Y(t)}{\sqrt{\Theta}} \right). \tag{5}$$

Proof. For any realization, multiplication of the state equation by the constant Z gives $\dot{q} = F - R$. Nonnegativity fixes the square-root sign, so $Y = bq$. Integration yields the identity

in (4). Strict survival gives $R(t) < \Theta$ everywhere. Continuity on the compact interval makes the maximum attained, hence $\Theta > R_{\mathrm{p}}$.

Conversely, choose $b \in \mathcal{K}(F, R)$ and $\Theta > R_{\mathrm{p}}$. Since B is absolutely continuous, the identity in (4) makes Y absolutely continuous as well. The state and parameters in (5) have $0 \le x < 1$, satisfy $GZx^2 = R$, and obey

$$\dot{x} = \frac{b(F - R)}{\sqrt{\Theta}} = \frac{F}{Z} - Gx^2 \tag{6}$$

almost everywhere. Every realization determines its own positive b and Θ by (2), so this construction is exhaustive. If survival is not imposed, the same converse works for every $\Theta > 0$. □

No division by Y or differentiation of the given output is required. The theorem therefore includes zero-output points. Assuming only that R is absolutely continuous and then differentiating $\sqrt{R}$ would not justify the same converse without further care.

Proposition 2 (Record-specific excitation and exceptions)**.** *If B is nonconstant, $\mathcal{K}$ has at most one element. For a compatible record,*

$$b = \frac{Y(t) - Y(s)}{B(t) - B(s)} \quad \text{whenever } B(t) \neq B(s), \qquad q(t) = \frac{Y(t)}{b}. \tag{7}$$

If B is constant, compatibility holds exactly when $R \equiv r$ and $F = r$ almost everywhere, for some $r \ge 0$. Then $\mathcal{K} = (0, \infty)$.

Proof. Subtract the identity in (4) at two times. Any nonzero denominator fixes b, which must be positive and satisfy the identity at all remaining times. If B is constant, $B(0) = 0$ makes it identically zero; compatibility requires constant Y, hence constant R, and $\dot{B} = F - R = 0$ almost everywhere. The converse is immediate. □

Thus an excited compatible record fixes $G/Z = b^2$ and $q(0)$ without prior inventory information. The word excited here means nonconstant B for this exact record. Constant forcing can suffice when the initial state is away from equilibrium. At positive equilibrium $q = \sqrt{r}/b$ remains unknown. For the identically zero record $q = x = 0$ for every positive parameter choice.

The known rescaling $(Z, G, x) \mapsto (\kappa Z, \kappa G, x/\kappa)$ preserves b, q and R, while sending Θ to $\kappa^2\Theta$. Equation (5) supplies the converse that a displayed rescaling alone would leave unproved.

4 Sharp critical-rate and future-contact information

Corollary 3 (Critical-rate identified set)**.** *For every compatible record with certified strict past survival, the set of possible critical rates is exactly*

$$\Theta \in (R_{\mathrm{p}}, \infty). \tag{8}$$

For an excited record, the corresponding capacity set is $Z \in (q_{\max}, \infty)$, where $q_{\max} = \max_I Y/b$, and $G = b^2 Z$.

The result is sharp: every point of the set is realized by Theorem 1. There is no finite upper bound and the lower endpoint is excluded. Without the survival label, the critical-rate set is all of $(0, \infty)$. On an open or infinite observation interval, a pointwise strict bound can coexist with an unattained supremum equal to Θ; the compact-interval hypothesis cannot be omitted.

For an excited record, prescribe a nonnegative measurable locally bounded future input on $[T, H]$. The common formal output continuation solves

$$\dot{Y} = b(F - Y^2), \qquad Y(T) = \sqrt{R(T)}. \tag{9}$$

This scalar problem has a unique nonnegative finite solution on finite intervals. Define

$$P_H = \max_{T \le t \le H} Y(t)^2, \qquad \tau_H(\Theta) = \inf\{t \in [T, H] : Y(t)^2 \ge \Theta\}, \tag{10}$$

with $\inf \emptyset = \infty$. The continuation is an auxiliary mathematical object. For each selected realization, its interpretation within the admitted model ends at its own first contact.

Theorem 4 (Exact finite-horizon decisions)**.** *If $P_H \le R_\mathrm{p}$, every past-compatible surviving realization stays strictly subcritical through H. If $P_H > R_\mathrm{p}$, exactly the thresholds $\Theta \in (R_\mathrm{p}, P_H]$ make contact by H, while $\Theta > P_H$ survive. No finite bounded-input future horizon forces contact for every uncalibrated past-compatible realization.*

Proof. At T, $Y(T)^2 \le R_\mathrm{p} < \Theta$. Continuity implies contact by H exactly when the maximum P_H is at least Θ. Intersect this condition with (8). The finite maximum leaves arbitrarily large compatible thresholds above it. □

Past survival can consequently certify a future interval even without identifying the critical rate. The relevant bound is the largest *past* output, not only the terminal output. The complete finite contact-time set on $[T, H]$ can also be written

$$\left\{ t \in (T, H] : Y(t)^2 > R_\mathrm{p} \text{ and } Y(s)^2 < Y(t)^2 \text{ for every } s \in [T, t) \right\}. \tag{11}$$

Indeed any first contact is the first attainment of its level above the past maximum; conversely set $\Theta = Y(t)^2$ at any time in (11). Later returns to an old peak and later points of a plateau cannot be first contacts with that level. A no-contact outcome remains possible on every finite horizon.

For an independently calibrated nonempty threshold set $C \subset (R_\mathrm{p}, \infty)$, all candidates survive exactly when every $\theta \in C$ exceeds P_H, and all contact exactly when every $\theta \in C$ is at most P_H. Otherwise both outcomes occur. This set formulation preserves open endpoints: $\inf C = P_H$ alone does not imply possible contact if that infimum is excluded.

If a locally bounded input is prescribed on all of $[T, \infty)$, the infinite-horizon statement has different quantifiers. If the formal future output is unbounded, each finite Θ is eventually reached. This supplies no common finite deadline over the unbounded threshold class. If the supremum is finite and approached only asymptotically, equality with that supremum need not cause finite-time contact.

5 Independent calibration and an analytic example

For an excited record, the remaining freedom is one-dimensional. A single independent scalar is sufficient precisely when it distinguishes all members of that class. Examples are

$$Z = z: \quad \Theta = b^2 z^2, \quad G = b^2 z, \tag{12}$$

$$G = g: \quad \Theta = g^2/b^2, \quad Z = g/b^2, \tag{13}$$

$$x(t_*) = \xi > 0: \quad \Theta = R(t_*)/\xi^2. \tag{14}$$

Each candidate must still satisfy $\Theta > R_\mathrm{p}$. Knowledge of $q(t_*)$, already reconstructed, does not remove this freedom. Neither does knowledge of the internal Jacobian magnitude $2Gx = 2bY$.

A zero-state calibration at zero output is similarly uninformative. A certified boundary contact outside the original strictly surviving interval, or in a separate endpoint-inclusive experiment, gives $\Theta = R(t_*)$ in a model valid to contact. Certification of that event is extra information, not a consequence of the smooth rate equation.

Stationary records require care. Known Z alone leaves G and Θ undetermined. A known positive x fixes Θ directly even at stationarity, but not b; a known positive q instead fixes $b = \sqrt{r}/q$, but not Θ. A change in forcing then has no unique rate prediction until b is known. For $r = 0$, knowing $q = x = 0$ identifies neither positive parameter.

A finite-pulse illustration. Use dimensionless units only for this mathematical example. On $0 \le t \le 1$, let

$$F(t) = 0, \quad R(t) = (1+t)^{-2}, \quad B(t) = -\frac{t}{1+t}, \quad Y(t) = \frac{1}{1+t}. \tag{15}$$

The identity in (4) gives $b = 1$ and $q = Y$, while strict past survival gives $\Theta > 1$. At $T = 1$ the output is only $1/4$, so a terminal-value bound would discard real past information.

Apply the constant future input $F = 4$ for a duration h. The exact common continuation is

$$Y(1+h) = 2\tanh\left(2h + \operatorname{artanh}\frac{1}{4}\right). \tag{16}$$

Every compatible threshold is safe until and including

$$h_0 = \frac{1}{2}\left(\operatorname{artanh}\frac{1}{2} - \operatorname{artanh}\frac{1}{4}\right), \tag{17}$$

when the output first recovers its past maximum 1. For a particular $1 < \Theta < 4$, contact during the constant input occurs at

$$h(\Theta) = \frac{1}{2}\left[\operatorname{artanh}\frac{\sqrt{\Theta}}{2} - \operatorname{artanh}\frac{1}{4}\right]. \tag{18}$$

If the pulse ends before this time and the input is then zero, the output decreases and no later contact occurs. Thus a pulse can exceed the stationary critical supply without reaching the boundary. At $\Theta = 4$, sustained input approaches the boundary only asymptotically; at larger thresholds it remains below it. This example uses an exact elementary solution and no fitted data or numerical experiment.

6 Physical scope and interpretation

The source family offers useful semantics but no calibrated rate experiment. An earlier note describes opposite energy-momentum transfer terms for matter and an Apeiron sector [7, sec. 1]. This is an intended internal-transfer interpretation, not a specified escaping-energy or luminosity law. The same note proposes accounting for ordinary accretion channels before interpreting any residual as conversion. It does not supply the independently measured histories granted here.

The later canonical interpretation explicitly treats reconversion as an internal organization change of a *single* carrier with conserved total stress energy [9]. Its local organization variable s and its directed evolution law do not define the horizonless stability coordinate x or derive (1). The missing item is therefore quantitative closure of this specific rate/state model, not the absence of a stated conservation principle. Internal conversion must not be counted automatically as destruction of total carrier energy or as escape to infinity.

The technical gravastar note defines $x = 0$ at activation and $x = 1$ at marginal stability, but leaves the map to a physical coordinate open [8, secs. 3,7]. Its reference model [5, secs. II.A–IV] provides a static density/compactness family and a radial pulsation eigenproblem. That result is not a derivation of the dissipative evolution in (1). In particular, a difference between two masses along an equilibrium family is not automatically the mass of a spatial reservoir in one evolving object.

The bookkeeping restriction to constant Z is essential. If $Z(t)$ varies and $q = Z(t)x$, the unmodified source equation instead gives $\dot{q} = F - R + x\dot{Z}$. Enforcing $\dot{q} = F - R$ would require an additional term $-x\dot{Z}/Z$ in the state equation. We do not insert it or analyze an alternative evolving-zone model here.

At a positive stationary point, $R = F$ and $\dot{q} = 0$. These equalities concern throughput in the derived inventory; they do not fix total gravitational-mass growth or the fate of conversion energy. Neither a measured relaxation time nor an observational compactness correlation follows without a physical transfer relation. A constant change of clock units rescales both rates and their inferred rate parameters; an unidentified model-to-observer clock cannot be absorbed while claiming a physically calibrated threshold. No horizon-avoidance statement follows from the scalar domain condition.

7 Discussion

Within the declared ideal experiment, nonstationary rates determine more than the original rescaling witness alone states: they determine the initial inventory and the full compatible class. Conversely, critical capacity remains free even when inventory reconstruction is exact. The surviving-history constraint supplies a sharp one-sided threshold set and sometimes a robust future safety statement. It never supplies an upper capacity bound on a finite record without independent information.

These results are a complete elementary specialization of known reservoir and identifiability methods. Their value is a precise accounting of what would have to be calibrated before a critical-reconversion boundary claim could be inferred from rate histories. They do not establish a new inverse methodology, a measured reconversion signal, or a completed physical compact-object model. Statistical recovery with noisy samples and a derived physical observation map are separate questions not settled by the exact identities.

A Regularity and zero-output details

The scalar right-hand side of (1) is measurable in time and locally Lipschitz in x. Nonnegative input makes the zero boundary inward-pointing. On each finite interval, $\dot{x} \le F/Z$ bounds the nonnegative solution and excludes positive finite-time blow-up. The same reasoning applies to (9).

For a compatible record q obeys $\dot{q} = F - b^2q^2 \ge -b^2q^2$. If $q(s) > 0$, scalar comparison gives

$$q(t) \ge \frac{q(s)}{1 + b^2q(s)(t-s)} > 0 \qquad (t \ge s) \tag{19}$$

on every finite subsequent interval. A positive output therefore cannot vanish in finite forward time under the assumptions. Initial zero output, a zero-input interval and the identically zero record are permitted. The compatibility identity automatically enforces the applicable restrictions; a square-root derivative at zero is unnecessary.

The positivity restriction also matters for exhaustiveness. If negative x were allowed, $Y = bq$ would fail because the output contains x^2. The theorem concerns exactly the nonnegative activation domain, not that larger signed-state inverse problem.

References

[1] J. W. Kirchner, Catchments as simple dynamical systems: Catchment characterization, rainfall-runoff modeling, and doing hydrology backward, *Water Resources Research* **45**, W02429 (2009), doi:10.1029/2008WR006912.

[2] A. F. Villaverde and G. Massonis, On testing structural identifiability by a simple scaling method: Relying on scaling symmetries can be misleading, *PLoS Computational Biology* **17**, e1009032 (2021), doi:10.1371/journal.pcbi.1009032.

[3] G. Cedersund, Conclusions via unique predictions obtained despite unidentifiability: new definitions and a general method, *FEBS Journal* **279**, 3513–3527 (2012), doi:10.1111/j.1742-4658.2012.08725.x.

[4] J. Lerat, Technical note: Quadratic Solution of the Approximate Reservoir Equation (QuaSoARe), *Hydrology and Earth System Sciences* **29**, 2003–2021 (2025), doi:10.5194/hess-29-2003-2025.

[5] D. Horvat, S. Ilijić and A. Marunović, Radial stability analysis of the continuous pressure gravastar, *Classical and Quantum Gravity* **28**, 195008 (2011), doi:10.1088/0264-9381/28/19/195008; arXiv:1104.3537, version 3 (2012).

[6] Apeiron project, *Horizonless Critical Reconversion*, authoritative v3.1, unpublished two-page model note (17 September 2026). Canonical equations, sec. 2; corrected family benchmark, sec. 3; stated open physical requirements, sec. 7.

[7] Apeiron project, *Horizonless Conversion / Missing Accretion Channel*, frozen research snapshot v1.0, unpublished two-page note (17 September 2026).

[8] J. Fellner, *Kritische horizonlose Rückwandlung: Kurze technische Diskussionsnotiz zu Horvat, Ilijić & Marunović (2011)*, unpublished two-page discussion note (17 September 2026).

[9] Apeiron project, *BH Reconversion: Scientific Feasibility Closure* and *Single-Carrier Consistency Addendum*, canonical thermodynamics update v1.4, reconversion texts v1.1 (19 September 2026), unpublished source notes. The closure text retains its original internal date of 14 September 2026.